Introduction to Ando - Haagerup theory

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Contents

Section 1 Introduction

§1.1 Main thm

Let M_n, φ_n be sequences of σ-fin. vN algs & faithful normal states on M_n.

Then we have

$$\sigma_t^{\varphi^\omega}\left((x_n)^\omega\right) = \left(\sigma_t^{\varphi_n}(x_n)\right)^\omega$$

for all $(x_n)^\omega \in (M_n, \varphi_n)^\omega$

$\uparrow$ Ocneanu ultraproduct.

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This result is due to
- Kirchberg
- Raynaud
- Ando - Haagerup

independently.

How can we prove that?

- One may use KMS condition? **(Raynaud)**

$\rightsquigarrow$ seems difficult since

$\alpha : \mathbb{R} \curvearrowright M$ continuous

$\wr$

$\alpha^\omega : \mathbb{R} \curvearrowright M^\omega$ discontinuous in general.

- Instead, use another kind of ultraproduct $\prod^\omega M_n$, the Groh - Raynaud ultraproduct.

Advantage:

- We can use all bdd sequences.
- We can "compute" the standard Hilbert space of $\prod^\omega M_n$.

In this note, we will treat a single M & its ultrapowers M^ω, $\prod^\omega M$. to simplify notations

§1.2

Notations

- M : vN alg. σ-finite

  math roman font.

- For $\varphi \in M_*^+$ & $x \in M$,

  $$\|x\|_\varphi := \varphi(x^*x)^{\frac{1}{2}}$$

  σ-strong top.

  $$\|x\|_\varphi^\# := \left(\varphi(x^*x) + \varphi(xx^*)\right)^{\frac{1}{2}}$$

  σ-strong* top.

- $\mathcal{Z}(M) := M' \cap M$

§1.3 ω

Let ω be a free ultrafilter on $\mathbb{N}$,

i.e.

- filter

$$\begin{cases} \cdot\ \phi \notin \omega \\ \cdot\ A, B \in \omega \Rightarrow A \cap B \in \omega \\ \cdot\ A \in \omega \ \& \ A \subset B \Rightarrow B \in \omega \end{cases}$$

- ultra

$${}^{\forall} A \in 2^{\mathbb{N}}, \text{ either } A \text{ or } A^c \in \omega$$

- free

ω is not of the form

$$\{A \in 2^{\mathbb{N}} \mid \underset{\text{given}}{\underset{\uparrow}{n}} \in A\}$$

$\Leftrightarrow {}^{\forall} A \in 2^{\mathbb{N}}$ finite $\quad A \notin \omega$

- $x_n \in \underset{\text{top sp.}}{X}$ sequence.

$x_n \to x \ , n \to \omega$

$\overset{\text{defn}}{\Longleftrightarrow} {}^{\forall} U \ni x$ open ${}^{\exists} A \in \omega$ s.t. $\underbrace{n \in A \Rightarrow x_n \in U}$

$$\underset{\omega}{\underset{\Uparrow}{A}} \subset \{n \mid x_n \in U\}$$

$\Leftrightarrow {}^{\forall} U \ni x$ open, $\{n \mid x_n \in U\} \in \omega$

~~filter~~

Thm. 1.1

${}^{\forall} x_n \in \underset{(\text{cpt Haus.})}{X} \quad (n \geq 1)$

${}^{\exists !} x \in X$ s.t. $x_n \to x \ (n \to \omega)$

proof

Otherwise, ${}^{\forall} y \in X$. ${}^{\exists} V_y$ open nbd of y.

$\{n \mid x_n \in V_y\} \notin \omega$. ...

Section 2 Ultraproduct vNalgs

§2.1 Why ultraproduct ?

Origin

$x_n \in \overset{\swarrow}{M}$ finite vNalg.

$\uparrow$ central sequence

(i.e. $[x_n . y] \xrightarrow[n\to\infty]{s^*} 0$ forall $y \in M$).

Let

$$x := (x_1, x_2, \cdots) \in \ell^\infty(M)$$

$$y := (y, y, \cdots) \in \ell^\infty(M).$$

$$\triangledown$$

$$I_\infty(M)$$

$$\shortparallel$$

$$\{(z_n) \mid z_n \xrightarrow[n\to\infty]{s^*} 0\}$$

$x_n y \sim y x_n$

$\downarrow$

Then $xy = yx$ in ℓ^∞ / I_∞

really equal $\rightsquigarrow$ various benefits.

But ℓ^∞ / I_∞ is NOT a vN alg.

$\rightsquigarrow$ consider I_ω

3 kinds of ultraproducts.

(1) Ocneanu ultraproduct §2.2.

$$M^\omega = \text{“} \ell^\infty(M) / I_\omega \text{”}$$

(2) Groh - Raynauld ultraproduct §2.3

$$\prod^\omega M \underset{\uparrow}{\subset} B(\prod^\omega H)$$

acts diagonally.

(3) Golodets type

$$R \cong M^\omega$$

So we don't treat this type.

Notations

M : fixed σ-fin. vNalg
possibly non-sep.

$$\ell^\infty := \ell^\infty(M)$$

$$= \{ (x_n) \mid x_n \in M \ \& \ \sup_{n \geqq 1} \|x_n\| < \infty \}$$

a unital C^*-alg

$$\mathcal{L}_\omega := \{ (x_n) \in \ell^\infty \mid x_n \xrightarrow{s} 0,\ n \to \omega \}$$

$$= \{ (x_n) \in \ell^\infty \mid \|x_n\|_\varphi \underset{n \to \omega}{\longrightarrow} 0 \}$$

φ : a faithful normal state on M

a norm closed left ideal of ℓ^∞

$$\mathcal{L}_\omega^* := \{ (x_n^*) \in \ell^\infty \mid (x_n) \in \mathcal{L}_\omega \}$$

a norm closed right ideal of ℓ^∞

$$\mathcal{I}_\omega := \mathcal{L}_\omega \cap \mathcal{L}_\omega^* \quad (\omega\text{-trivial sequence})$$

$$= \{ (x_n) \in \ell^\infty \mid x_n \xrightarrow[n \to \omega]{s^*} 0 \}$$

a hereditary C^*-subalg of ℓ^∞.

(not an ideal of ℓ^∞ in general
it is if M finite.
because $x_n \xrightarrow{s^*} 0 \iff x_n \xrightarrow{s} 0$)

$$\mathcal{M}^\omega := \{ \underset{x}{\cancel{(x_n)}} \in \ell^\infty \mid x \mathcal{I}_\omega \subset \mathcal{I}_\omega,\ \mathcal{I}_\omega x \subset \mathcal{I}_\omega \}$$

a unital C^*-subalg of ℓ^∞.

(normalizer or idealizer or multiplier of $\mathcal{I}_\omega$ in ℓ^∞).

§2.2 Ocneanu ultraproduct

Defn. 2.1

$$M^\omega := \mathcal{M}^\omega / \mathcal{I}_\omega \quad \text{quotient } C^*\text{-alg}$$

$$(x_n)^\omega := (x_n) + \mathcal{I}_\omega$$

Thm 2.2 (Sakai, Ocneanu, AH)
finite separable non-sep.

(1) M^ω is a W^*-alg.

(2) For $\forall \varphi \in M_*$, let

$$\varphi^\omega((x_n)^\omega) := \lim_{n\to\omega} \varphi(x_n)$$

for $(x_n)^\omega \in M^\omega$.

Then $\varphi^\omega \in (M^\omega)_*$

proof

It will follow from Prop 3.8(2).

So, we treat M^ω as a C^*-alg in what follows.

✡ If $\varphi \in M_*^+$ faithful,

then $\varphi^\omega \in (M^\omega)^*$ is a faithful state.

Indeed

$$\|(x_n)^\omega\|_{\varphi^\omega} \overset{\text{by defn. of } \varphi^\omega}{=} \lim_{n\to\omega} \|x_n\|_\varphi$$

~~If $(x_n)^\omega \geq 0$ &~~

Thus which $= 0 \iff |x_n| \xrightarrow[n\to\omega]{s} 0$

$\Rightarrow$

Then $\underset{\underset{0}{\swarrow s}}{x_n} \underset{\underset{\mathcal{M}^\omega}{\in}}{x_n^*} \xrightarrow[n\to\omega]{s} 0$ i.e. $x_n^* \xrightarrow[n\to\omega]{s} 0$

$\Bigg\}\ (x_n)^\omega = 0$

✩ If M is not finite, then

$$\mathcal{M}^{\omega} \subsetneqq \ell^{\infty}.$$

(use the shift operator).

§2.3 Groh-Raynaud ultraproduct

- Ultraproduct Hilbert space.

Let H a Hilb. sp.

$$\ell^\infty(H) := \{ (\xi_n) \mid \sup_{n\geq 1} \|\xi_n\| \leq \infty \}$$ Banach sp.

$$\cup$$

$$\mathcal{J}_\omega := \{ (\xi_n) \in \ell^\infty \mid \lim_{n\to\omega} \|\xi_n\| = 0 \}.$$

closed subsp.

Then define

$$H_\omega := \ell^\infty(H) / \mathcal{J}_\omega ,$$

which is a quotient Banach sp.

$$(\xi_n)_\omega := (\xi_n) + \mathcal{J}_\omega$$

Easy to check

$$\underset{\substack{\uparrow \\ \text{quotient norm}}}{\|(\xi_n)_\omega\|} = \lim_{n\to\omega} \|\xi_n\|$$

quotient norm.

- $\prod^\omega M$.

Let $M \subset B(H)$.

Then we have

$$\ell^\infty \xrightarrow[\pi_\omega]{} B(H_\omega) \quad \text{*-homo.}$$

by

$$\pi_\omega((x_n))(\xi_n)_\omega := (x_n \xi_n)_\omega$$ well-def.

"diagonal action"

We will write

$$(x_n)_\omega := \pi_\omega((x_n)).$$

Hence

$$\boxed{\begin{array}{l}(x_n)_\omega(\xi_n)_\omega = (x_n\xi_n)_\omega \\ \text{for } (x_n) \in \underset{\substack{\uparrow \\ \text{all norm bdd sequences !!}}}{\ell^\infty} \ \& \ (\xi_n)_\omega \in H_\omega\end{array}}$$

<u>Defn</u>. 2.3. (Groh-Raynaud ultraproduct)

$$\prod^\omega M := \overline{\{(x_n)_\omega \mid (x_n) \in \ell^\infty\}}^{\sigma\text{-w}}$$

$$\subset B(H_\omega)$$

⌟

<u>Lem</u>. 2.4.

$$\|(x_n)_\omega\| = \lim_{n\to\omega} \|x_n\|$$

<u>proof</u>

Let $(\xi_n)_\omega \in H_\omega$ $\|(\xi_n)_\omega\| \leqq 1$.

WMA $\|\xi_n\| \leqq 1$ for all n.

Then

$$\|(x_n)_\omega(\xi_n)_\omega\| = \|(x_n\xi_n)_\omega\|$$

$$= \lim_{n\to\omega} \|x_n\xi_n\|$$

$$\leqq \lim_{n\to\omega} \|x_n\|.$$

$\therefore$ LHS $\leqq$ RHS.

Next take $\xi_n \in H$, $\|\xi_n\| \leqq 1$.

s.t.

$$\|x_n\| - \frac{1}{n} \leqq \|x_n\xi_n\|$$

Then for $\xi := (\xi_n)_\omega$,

$$\begin{array}{ccc}\|(x_n)_\omega\xi\| & \leqq & \|(x_n)_\omega\| \\ \| & & \\ \lim_{n\to\omega} \|x_n\xi_n\| & \geqq & \lim_{n\to\omega}\left(\|x_n\| - \frac{1}{n}\right)\end{array}$$

$\therefore$ LHS $\geqq$ RHS ∎

? Show $\Pi^{\omega} M$ depends on H.

Section 3 M^ω vs $\prod^\omega M$

Recall

$$M^\omega \curvearrowright L^2(M^\omega, \varphi^\omega)$$

the GNS space wrt φ^ω

$\varphi \in M_*^+$ faithful state

and

$$\prod^\omega M \curvearrowright L^2(M, \varphi)_\omega .$$

For $(x_n) \in \mathcal{M}^\omega \subset \ell^\infty$, we have

$$\|(x_n)^\omega \xi_{\varphi^\omega}\|^2 = \varphi^\omega((x_n^* x_n)^\omega)$$

$$= \lim_{n\to\omega} \varphi(x_n^* x_n)$$

$$= \lim_{n\to\omega} \|x_n \xi_\varphi\|^2$$

$$= \|(x_n)_\omega (\xi_\varphi)_\omega\|^2$$

Thus we have an isometry

$$\begin{tikzcd} w_\varphi : L^2(M^\omega, \varphi^\omega) \arrow[r] & L^2(M, \varphi)_\omega \\ (x_n)^\omega \xi_{\varphi^\omega} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (x_n \xi_\varphi)_\omega \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

Our main thm in this section states

Thm 3.1

$$M^\omega = w_\varphi^* \prod^\omega M \, w_\varphi$$

☆ We can compute $w_\varphi w_\varphi^*$ in Section 4 (Thm 4.6)

§3.1. $\mathcal{L}^\omega$, $\mathcal{I}_\omega$, $\mathcal{M}^\omega$

Recall

$$\begin{tikzcd} \ell^\infty \arrow[d, phantom, "\cup"] & \\ \mathcal{L}_\omega \arrow[r, squiggly] & \mathcal{L}_\omega \cap \mathcal{L}_\omega^* = \mathcal{I}_\omega \arrow[d, phantom, "\wr"] \\ & \mathcal{M}^\omega = \text{the multiplier of } \mathcal{I}_\omega \text{ in } \ell^\infty \end{tikzcd}$$

Let us study this situation in a general setting.

A : C^*-alg

L : closed left ideal in A given.

$B := L \cap L^*$ hereditary C^*-subalg.

$M := \{a \in A \mid aB \subset B,\ Ba \subset B\}$ multiplier of B in A

(This M does not mean a vN alg!)
(Forget vN algs for a while.)

Thm 3.2.

$L = \{a \in A \mid a^*a \in B\}$

Proof See Dixmier. $\boxed{\text{E}}$

Lem 3.3.

(1) $LM \subset L$, $ML^* \subset L^*$

(2) $M \cap (L + L^*) = B$.

proof

(1) $x \in L,\ y \in M.$

$$(xy)^*(xy) = y^* \underset{\overset{\cap}{B}}{\underset{\sim}{x^*x}}\, y$$

$$\in B$$

(2) STEP I $M \cap L = B.$

$\supset$ trivial. since $\overset{\overset{L}{\cup}}{B} \subset M$

$\subset$ $x \in M \cap L.$

$x^*x \in L^*L \subset B.$

Since $x \in M$

$$\underset{\overset{\shortparallel}{(xx^*)^2}}{x(x^*x)x^*} \in B$$

Hence $xx^* = ((xx^*)^2)^{1/2} \in B.$

i.e. $x^* \in L$ by Thm 3.2.

$\Longleftrightarrow x \in L^*$

Thus $x \in L \cap L^* = B.$

STEP II

$$M \cap (L + L^*) = (M \cap L) + (M \cap L^*).$$

$\supset$ trivial.

$\subset$ $x = y + z^* \in LHS,$

$(y \in L,\ z \in L)$

Let $b \in B = L \cap L^*.$

Then $by \in L^*L \subset B.$

$$yb = \underset{\overset{\cap}{B}}{\underset{\sim}{\overset{}{\underset{\overset{\cap}{M}}{x}}b}} - \underset{\overset{\cap}{L^*L}}{z^*b} \in B$$

Hence $y \in M.$ Similary $z \in M$

∎

Prop 3.4

(1) $\mathcal{L}_\omega \mathcal{M}^\omega \subset \mathcal{L}_\omega$

$\mathcal{M}^\omega \mathcal{L}_\omega^* \subset \mathcal{L}_\omega^*$

(2) $\mathcal{M}^\omega \cap (\mathcal{L}_\omega + \mathcal{L}_\omega^*) = \mathcal{I}_\omega$ $\lrcorner$

§3.2 $M^\omega \cong$ corner of $\prod^\omega M$

Recall

M : σ-fin. vN alg.

φ : a faithful normal state on M.

Put

$H := L^2(M, \varphi)$.

$\xi_\omega := (\xi_\varphi)_\omega \in H_\omega$

$\varphi_\omega(x) := \langle x\xi_\omega, \xi_\omega \rangle$ for $x \in \prod^\omega M$.

Then φ_ω is a normal state on $\prod^\omega M$.

Defn. 3.5

$$p_\varphi := \text{support projection of } \varphi_\omega$$

$$= \text{proj} : H_\omega \longrightarrow \overline{(\prod^\omega M)'\xi_\omega}$$

$$\in \prod^\omega M$$

Lem. 3.6.

For $\forall x \in \prod^\omega M$, $\exists (x_n) \in \ell^\infty$, s.t.

(1) $x\xi_\omega = (x_n)_\omega \xi_\omega$

$x^*\xi_\omega = (x_n^*)_\omega \xi_\omega$

(2) $x - p_\varphi^\perp x p_\varphi^\perp = (x_n)_\omega - p_\varphi^\perp (x_n)_\omega p_\varphi^\perp$

Let us prove the above afterwards.

Prop. 3.7.

$N := \prod^{\omega} M$

$p := p_{\varphi} \in N$

$\exists$ vector space isomo.

$$\rho : \ell^{\infty}/\mathcal{I}_{\omega} \to pNp \oplus pNp^{\perp} \oplus p^{\perp}Np$$

such that.

$$\rho^{-1}(pNp) = \mathcal{M}^{\omega}/\mathcal{I}_{\omega} \quad (= M^{\omega})$$

$$\rho^{-1}(pNp^{\perp}) = \mathcal{L}^{\omega}/\mathcal{I}_{\omega}$$

$$\rho^{-1}(p^{\perp}Np) = \mathcal{L}^{\omega *}/\mathcal{I}_{\omega}.$$

In particular,

$$\ell^{\infty} = \mathcal{M}^{\omega} + \mathcal{L}_{\omega} + \mathcal{L}_{\omega}^{*}$$

$\lrcorner$

Proof

$(x_n) \in \mathcal{L}_{\omega}$

$\Leftrightarrow x_n \xrightarrow{s} 0, \quad n \to \omega$

$\Leftrightarrow \lim_{n \to \omega} \| x_n \xi_{\varphi} \| = 0 \qquad (\text{in } H = L^2(M, \varphi))$

$\| x_n \xi_{\varphi} \| = \| (x_n)_{\omega} \xi_{\omega} \|$

$\Leftrightarrow (x_n)_{\omega} \xi_{\omega} = 0$

$\Leftrightarrow (x_n)_{\omega} p = 0$

$(x_n) \in \mathcal{L}_{\omega}^{*} \Leftrightarrow p(x_n)_{\omega} = 0.$

Thus,

$(x_n) \in \mathcal{I}_{\omega} \Leftrightarrow (x_n)_{\omega} \in p^{\perp}Np^{\perp}.$

— (*)

Let us put

$$\rho\big(\underset{\underset{\ell^\infty/\mathcal{I}_\omega}{\rotatebox{90}{$\in$}}}{(x_n)+\mathcal{I}_\omega}\big) := \underset{\underset{pNp \oplus pNp^\perp \oplus p^\perp Np.}{\rotatebox{90}{$\in$}}}{(x_n)_\omega - p^\perp (x_n)_\omega p^\perp}$$

which is well-defined.

By $(*)$, ρ is injective.

By Lem. 3.6 (2),

$$\rho : \ell^\infty/\mathcal{I}_\omega \longrightarrow pNp \oplus pNp^\perp \oplus p^\perp Np$$

is surjective.

Then

$$(x_n)+\mathcal{I}_\omega \in \rho^{-1}(pNp^\perp)$$

since $pp^\perp = 0$

$$\Leftrightarrow (x_n)_\omega - p^\perp (x_n)_\omega p^\perp = p(x_n)_\omega p^\perp$$

$$\Leftrightarrow (x_n)_\omega = (x_n)_\omega p^\perp$$

$$\Leftrightarrow (x_n)_\omega p = 0$$

$$\Leftrightarrow (x_n) \in \mathcal{L}_\omega$$

Thus,

$$\rho^{-1}(pNp^\perp) = \mathcal{L}_\omega/\mathcal{I}_\omega$$

$$\rho^{-1}(p^\perp Np) = \mathcal{L}_\omega^*/\mathcal{I}_\omega$$

We will show

$$\rho^{-1}(pNp) = \mathcal{M}^\omega/\mathcal{I}_\omega.$$

"$\subset$"

$$(x_n)+\mathcal{I}_\omega \in \rho^{-1}(pNp)$$

Let $(y_n) \in \mathcal{I}_\omega$

i.e. $(y_n)_\omega \in p^\perp N p^\perp$ by $(*)$.

$$\rho((y_n x_n) + \mathcal{I}_\omega)$$

$$= (y_n x_n)_\omega - p^\perp (y_n x_n)_\omega p^\perp$$

$$= \underset{\underset{p^\perp N p^\perp}{\cap}}{(y_n)_\omega} \big(\underset{\underset{\rho(a)\, \in\, pNp}{\|}}{(x_n)_\omega - p^\perp (x_n)_\omega p^\perp} \big)$$

$$= 0 .$$

ρ injective $\rightsquigarrow$ $(y_n x_n) \in \mathcal{I}_\omega$.

Thus $(x_n) \in \mathcal{M}^\omega$.

"$\supset$" Let $(x_n) \in \mathcal{M}^\omega$.

$$\rho((x_n) + \mathcal{I}_\omega) = \underset{\underset{pNp}{\cap}}{a} + \underset{\underset{pNp^\perp}{\cap}}{b} + \underset{\underset{p^\perp Np}{\cap}}{c}$$

$$= \rho(\underset{\underset{\mathcal{M}^\omega}{\boxed{\cap}}}{(\mathring{a}_n)} + \underset{\underset{\mathcal{L}_\omega}{\cap}}{(\mathring{b}_n)} + \underset{\underset{\mathcal{L}_\omega^*}{\cap}}{(\hat{c}_n)} + \mathcal{I}_\omega)$$

By the proof of "$\subset$" $\longrightarrow$ $\boxed{\cap}$

Since ρ injective,

$$\underset{\underset{\mathcal{M}^\omega}{\cap}}{(x_n) - (\mathring{a}_n)} \in (\mathring{b}_n) + (\hat{c}_n) + \mathcal{I}_\omega$$

$$\subset \mathcal{L}_\omega + \mathcal{L}_\omega^* + \mathcal{I}_\omega$$

$$\subset \mathcal{L}_\omega + \mathcal{L}_\omega^*$$

Thus

$$(x_n) - (\mathring{a}_n) \in \mathcal{M}^\omega \cap (\mathcal{L}_\omega + \mathcal{L}_\omega^*)$$

$$\| \\ \mathcal{I}_\omega$$

by Prop. 3.4(2).

$\blacksquare$

Thus

$$\rho((x_n) + \mathcal{I}_\omega) = \rho((\mathring{a}_n) + \mathcal{I}_\omega)$$

$$= a$$

Prop 3.8

(1) $(x_n) \in \mathcal{M}^\omega \Longleftrightarrow p_\varphi (x_n)_\omega = (x_n)_\omega p_\varphi$

(2) $\rho|_{M^\omega} : M^\omega \longrightarrow p_\varphi (\prod^\omega M) p_\varphi$ is

a ~~W~~ C^*-alg isomorphism. $(x_n)^\omega \mapsto (x_n)_\omega p_\varphi$

In particular, M^ω is a W^*-alg. Thm 2.2(1)

proof

(1) By the previous prop,

$(x_n) + \mathcal{I}_\omega \in \mathcal{M}^\omega / \mathcal{I}_\omega$

$\Longleftrightarrow \rho((x_n) + \mathcal{I}_\omega) \in pNp$

$\Longleftrightarrow (x_n)_\omega - p^\perp (x_n)_\omega p^\perp = p(x_n)p$

$\Longleftrightarrow (x_n)_\omega = p(x_n)_\omega p + p^\perp (x_n)_\omega p^\perp$

$\Longleftrightarrow (x_n)_\omega p = p(x_n)_\omega .$

(2) By (1),

$$\rho((x_n)^\omega) = (x_n)_\omega - p^\perp (x_n)_\omega p^\perp$$
$$= \underbrace{(x_n)_\omega p}_{\text{commutative}}$$

Thus, $\rho|_{M^\omega}$ is a $*$-homo.

injectivity & surjectivity follows from the prev. prop. ▨

$$\langle \rho((x_n)^\omega) \xi_\omega, \xi_\omega \rangle = \langle (x_n)_\omega p \xi_\omega, \xi_\omega \rangle$$
$$= \lim_{n \to \omega} \langle x_n \xi_\varphi, \xi_\varphi \rangle$$
$$= \lim_{n \to \omega} \varphi(x_n)$$
$$= \varphi^\omega((x_n)^\omega) .$$

Thm 2.2(2)

$\varphi^\omega \in M_* \longleftarrow$

Proof of Thm 3.1

Recall

$$\begin{tikzcd} w_\varphi : L^2(M^\omega, \varphi^\omega) \arrow[r] & L^2(M, \varphi)_\omega \\ \end{tikzcd}$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x_n)^\omega \xi_{\varphi^\omega} & \longmapsto & (x_n \xi_\varphi)_\omega . \end{array}$$

Let

$N := \prod^\omega M$

$\xi_\omega := (\xi_\varphi)_\omega$

$P := P_\varphi$

$w := w_\varphi .$

Want to show

$$w^* N w = M^\omega .$$

" $\supset$ " Let

$(x_n) \in \mathcal{M}^\omega$ & $(y_n)^\omega \in M^\omega$.

Then

$$(x_n)_\omega \, w (y_n)^\omega \xi_{\varphi^\omega}$$

$$= (x_n)_\omega \, (y_n \xi_\varphi)_\omega$$

$$= (x_n y_n \xi_\varphi)_\omega$$

$$= w \, (x_n)^\omega (y_n)^\omega \xi_{\varphi^\omega}$$

Hence For $(x_n) \in \underline{\underline{\mathcal{M}}}^\omega$ (not ℓ^∞)

$$(x_n)_\omega \, w = w \, (x_n)^\omega ,$$

$$\begin{array}{c} \rotatebox{90}{$\in$} \\ N \end{array}$$

and $w^* (x_n)_\omega w = (x_n)^\omega$ — $(*)$

"$\subset$" $N = \overline{\{(x_n)_\omega \mid (x_n) \in \ell^\infty\}}^{\sigma\text{-}w}$.

Let $(x_n) \in \ell^\infty$.

Then by Prop. 3.7,

$$(x_n) = \underset{\mathcal{M}^\omega}{\overset{\cap}{(a_n)}} + \underset{\mathcal{L}_\omega}{\overset{\cap}{(b_n)}} + \underset{\mathcal{L}_\omega^*}{\overset{\cap}{(c_n)}}.$$

Thus,

$$w^*(x_n)_\omega w = \boxed{w^*(a_n)_\omega w} = (a_n)^\omega \in M^\omega \xleftarrow{} \text{by } (*)$$

$$+$$

$$\underset{\substack{\| \\ 0}}{w^*(b_n)_\omega w} + \underset{\substack{\| \\ 0}}{w^*(c_n)_\omega w}$$

want to prove

Let $(y_n)^\omega \in M^\omega$.

Then

$$(b_n)_\omega w (y_n)^\omega \xi_{\varphi^\omega}$$

$$= ~~(b_n y_n)_\omega~~ \xi_\omega$$

$$(\underset{\mathcal{L}_\omega}{\overset{\cap}{b_n}} \underset{\mathcal{M}^\omega}{\overset{\cap}{y_n}} \xi_\varphi)_\omega \rightsquigarrow b_n y_n \in \mathcal{L}_\omega$$

(by Lem 3.3 (1))

$$= 0$$

Hence $(b_n)_\omega w = 0$

Cor 3.9

For $\forall (a_n) \in \ell^\infty$,

$\exists\ (b_n) \in \mathcal{M}^\omega,\ (c_n) \in \mathcal{L}_\omega,\ (d_n) \in \mathcal{L}_\omega^*$

s.t.

(1) $a_n = b_n + c_n + d_n \quad \forall n$

(b_n) is unique up to $\mathcal{I}_\omega$

(2) $\|(b_n)^\omega\| \leqq \|(a_n)_\omega\|$

$$= \lim_{n\to\omega} \|a_n\|$$

⌟

proof.

By Prop. 3.7,

$$\ell^\infty = \mathcal{M}^\omega + \mathcal{L}_\omega + \mathcal{L}_\omega^*.$$

$$(a_n) = (b_n) + (c_n) + (d_n)$$

By prop 3.4(2), (b_n) is unique up to $\mathcal{I}_\omega$.

Since

$$\rho|_{M^\omega} : \begin{array}{ccc} M^\omega & \xrightarrow{\sim} & pNp \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x_n)^\omega & \longmapsto & (x_n)_\omega p \end{array},$$

$$\begin{aligned} \|(b_n)^\omega\| &= \|\rho((b_n)^\omega)\| \\ &= \|(b_n)_\omega p\| \\ &= \|p(b_n)_\omega p\| \\ &= \|p(a_n)_\omega p\|. && \text{by the proof of prop 3.7.} \\ &\leqq \|(a_n)_\omega\| \\ &= \lim_{n\to\omega} \|a_n\| && \text{by Lem 2.4.} \end{aligned}$$

▨

proof of Lem 3.6

Let $E \subset H_\omega \oplus H_\omega$ be as follws

$$E := \{ ((x_n)_\omega \xi_\omega, (x_n^*)_\omega \xi_\omega) \mid \sup_{n\geq 1} \|x_n\| \leq 1 \}$$

Claim E is a closed subset.

proof
$(\eta, \zeta) \in \overline{E}$

$\Rightarrow$ we can take $(x_n^k) = (x_1^k, x_2^k, \cdots) \in \ell^\infty$.

such ~~that~~

$$\begin{cases} \| (x_n^k)_\omega \xi_\omega - \eta \| < \frac{1}{2^{k+1}} \\ \| (x_n^{k*})_\omega \xi_\omega - \zeta \| < \frac{1}{2^{k+1}} \end{cases}$$

for each k,

i.e. for big ~~k~~ n's (close to ω)

$$\| x_n^{k+1} \xi_\varphi - x_n^k \xi_\varphi \| < \frac{1}{2^{k+1}}$$

$$\| x_n^{k+1*} \xi_\varphi - x_n^{k*} \xi_\varphi \| < \frac{}{2^{k+1}}$$

$$\begin{array}{llll} x_1^1 & x_2^1 & \cdots & x_n^1 \quad \cdots \\ x_1^k & x_2^k & \cdots & x_n^k \quad \cdots \end{array}$$

Take x_n^k "diagonally"

$P\omega(x_n)$.

Then we obtain

$$(x_n)_\omega \xi_\omega = \eta$$

$$(x_n^*)_\omega \xi_\omega = \zeta$$

Claim $\boxed{E}$

(1) Let $x \in N (= \Pi^\omega M)$

By Kaplanski, $\exists$ net $(x_n^\alpha) \in \ell^\infty$ $\|x_n^\alpha\| \leq \|x\|$

s.t.

$$(x_n^\alpha) \xrightarrow[\alpha]{s^*} x$$

$$\therefore (x \xi_\omega, x^* \xi_\omega) \in \overline{E} = E.$$

26 (2) Let $y \in N'$.

Take (x_n) for x as (1).

Then.

$$x y \xi_\omega = y x \xi_\omega$$
$$= y (x_n)_\omega \xi_\omega$$
$$= (x_n)_\omega y \xi_\omega$$

$\rightsquigarrow$ $x p = (x_n)_\omega p$

similary

$x^* p = (x_n^*)_\omega p$

i.e.

$p x = p (x_n)_\omega$

$\rightsquigarrow$ $x = \begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$

$(x_n)_\omega = \begin{array}{|c|c|} \hline A & B \\ \hline C & D' \\ \hline \end{array}$

∎

Section 4

Standard forms

§4.1 Standard form of $\Pi^{\omega}M$

Defn. 4.1

(M, H, J, P) standard form.

$$\begin{cases} \cdot\ M \curvearrowright H \\ \cdot\ J : H \to H \quad \text{antilinear unitary} \\ \qquad\qquad J^2 = 1 \\ \cdot\ P \subset H \quad \text{closed convex cone} \\ \qquad\qquad \text{self-dual.} \end{cases}$$

iff

(1) $JMJ = M'$

(2) $J\xi = \xi$ for $\xi \in P$

(3) $xJxJP \subset P$ $\forall x \in M$

(4) $JxJ = x^*$, $\forall x \in Z(M)$

Prop 4.2 (4) can be dropped.

proof (Sketch).

We reduce the problem to the case

$^{\exists}\xi \in P$ cyc & sep. wrt.

Then

$$\begin{cases} JMJ = M' \\ J\xi = \xi \\ \langle \xi, aJaJ\xi \rangle \geq 0 \quad {}^{\forall}a. \ \underbrace{\cancel{(\text{equality} \Leftrightarrow a = 0)}}_{\text{不要}} \end{cases}$$

$\underset{(*)}{\rightsquigarrow} J = J_\xi$

Araki's characterization

$\sim J_\xi \, x \, J_\xi = x^* \qquad x \in Z(M).$

Tomita–Takesaki.

$\blacksquare$

About (*).

$$S_\xi = J_\xi \Delta_\xi^{\frac{1}{2}}$$

Then

$$\langle J S_\xi a\xi, a\xi \rangle$$

$$= \langle J \underset{JJ}{a^*} \xi, a\xi \rangle$$

$$= \langle \xi, aJaJ\xi \rangle \geq 0. \quad (2', (3)$$

JS_ξ is closed operator with core $M\xi$

which is symmetric & positive.

$$\longrightarrow (JS_\xi) \subset (JS_\xi)^*$$

~~Once~~ Want to show. JS_ξ is self-adj.

$$(JS_\xi)^* = S_\xi^* J$$

By Tomita-Takesaki,

$$D(S_\xi^*) \underset{\text{core}}{\supset} M'\xi \overset{(1)}{=} JM\xi$$

$$\qquad\qquad\qquad\qquad \cup\!\!\!\!\!\!\; D(JS_\xi)$$

$$\therefore \quad D((JS_\xi)^*) \underset{\text{core}}{\supset} M\xi \qquad \square$$

Now let $(M.H.J.P)$ standard.

- $H_\omega : \cancel{= \ell(H)_\omega}$ as before.
- $J_\omega (\xi_n)_\omega := (J\xi_n)_\omega$
- $P_\omega := \{ (\xi_n)_\omega \mid \xi_n \in P \}$

Thm. 4.3

$(\cancel{M^\omega}\ \Pi^\omega M, H_\omega, J_\omega, P_\omega)$ is standard.

Trivials

- $J_\omega \Pi^\omega M J_\omega = \Pi^\omega M' \subset (\Pi^\omega M)'$
- $J_\omega \xi = \xi \quad \forall \xi \in P_\omega$
- $x J_\omega x J_\omega P_\omega \subset P_\omega \quad \forall x \in \Pi^\omega M$

must show that

- P_ω is closed, selfdual, cone
- $(\Pi^\omega M)' = \Pi^\omega M'$

proof

- P_ω cone, trivial
- closed?

In general $X \subset E$ closed set

$(X)_\omega \subset (E)_\omega$

$\uparrow$ closed. use the diagonal argument.

- self-dual?

$$\xi \in P_\omega^0$$
$$\|$$
$$(\xi_n)_\omega \qquad \xi_n \in H.$$
$$\|$$
$$\eta_n^+ - \eta_n^- + i(\zeta_n^+ - \zeta_n^-)$$

- $\eta_n^\pm, \zeta_n^\pm \in P$
- $\langle \eta_n^+, \eta_n^- \rangle = 0. \ \langle \zeta_n^+, \zeta_n^- \rangle = 0.$

$$0 \leq \langle \overset{\mathcal{P}_\omega}{\xi}, (\eta_n^-)_\omega \rangle = \lim_{n\to\omega} \langle \xi_n, \eta_n^- \rangle$$

$$= -\lim_{n\to\omega} \|\eta_n^-\|^2 + i \lim_{n\to\omega} \langle \underset{\underset{\mathbb{R}}{\Uparrow}}{\zeta_n^+ - \zeta_n^-}, \eta_n^- \rangle$$

$$\therefore \quad (\eta_n^-)_\omega = 0$$

$$0 \leq \langle \xi, (\zeta_n^\pm)_\omega \rangle = \lim_{n\to\omega} \langle \xi_n, (\zeta_n^\pm) \rangle$$

$$= \lim_{n\to\omega} (\langle \xi_n^+, \zeta_n^\pm \rangle \pm i \underset{\underset{\mathbb{R}}{\Uparrow}}{\|\zeta_n^\pm\|^2})$$

$$\therefore \quad (\xi_n^{\pm})_\omega = 0$$

$$\therefore \quad \xi = (\xi_n^+)_\omega \in \mathcal{P}_\omega .$$

▨

To prove $(\Pi^\omega M)' = (\Pi^\omega M')$ we need the follwg lema.

Lem 4.4

may not be standard

$M \subset B(H)$

$\forall \xi \in H_\omega$. $\forall a' \in (\Pi^\omega M)' \curvearrowright H_\omega$

$\exists\ a \in \Pi^\omega M'$

s.t. $a'\xi = a\xi$

proof

WMA $\|a'\| = 1$

Put $\eta := a'\xi$. , $\xi = (\xi_n)_\omega$

$\eta = (\eta_n)_\omega$

Let

$$\varepsilon_n := \sup\{\langle x\eta_n, \eta_n\rangle - \langle x\xi_n, \xi_n\rangle \mid x \in M,\ 0 \leqq x \leqq 1\}.$$

Then $\varepsilon_n \geqq 0$. since $0 \in M$.

By weak compactness of $\{x \mid 0 \leqq x \leqq 1\}$,

$\exists\ x_n \in M$ s.t. $0 \leqq x_n \leqq 1$

&

$\varepsilon_n = \langle x_n \eta_n, \eta_n\rangle - \langle x_n \xi_n, \xi_n\rangle$.

Then

$$\lim_{n\to\omega} \varepsilon_n = \langle (x_n)_\omega \eta, \eta\rangle - \langle (x_n)_\omega \xi, \xi\rangle$$

commute.

$$= \langle (x_n)_\omega \underbrace{(a'^* a' - 1)}_{\leqq 0}\ \xi, \xi\rangle$$

$$\leqq 0.$$

Thus

$$\lim_{n\to\omega} \varepsilon_n = 0. \qquad \text{---} (*).$$

Now put

$$\varphi_n := \omega_{\eta_n} - \omega_{\xi_n} \in M_*.$$

Let

$$\varphi_n = \varphi_n^+ - \varphi_n^-$$

be the Jordan decomposition.

$P_n :=$ support proj. of φ_n^+

Then $\varphi_n^-(P_n) = 0$, and.

$$\|\varphi_n^+\| = \varphi_n^+(P_n) = \varphi_n(P_n)$$

$$= \omega_{\eta_n}(P_n) - \omega_{\xi_n}(P_n)$$

$$\leqq \varepsilon_n.$$

Thus

$$\lim_{n \to \omega} \|\varphi_n^+\| = 0 \quad \text{by } (*)$$

Next,

$$\omega_{\eta_n} + \varphi_n^- = \omega_{\xi_n} + \varphi_n^+,$$

and

$$\omega_{\eta_\eta} \leqq \omega_{\xi_n} + \underset{\uparrow}{\underset{\sim}{\varphi_n^+}}.$$

very small as n → ω

<u>Claim.</u>

M : vN alg $\subset B(H)$ may not be std.

Suppose for some $\xi, \eta \in H$ & $\varphi \in M_*^+$,

$$\omega_\eta \leqq \omega_\xi + \varphi.$$

Then $\exists\ x' \in M'$ & $\zeta \in H$

s.t.
- $\|x'\| \leqq 1$
- $\eta = x'\xi + \zeta$
- $\|\zeta\| \leqq \|\varphi\|^{1/2}$

Proof of Claim

Take $\zeta_n \in H$ $n \geq 1$. s.t.

$$\varphi = \sum_{n=1}^{\infty} \omega_{\zeta_n}$$

Let $\widetilde{H} = H \otimes \ell^2(\mathbb{Z}_{\geq 0})$

$$\widetilde{\eta} = \eta \otimes \varepsilon_0$$

$$\widetilde{\xi} = \xi \otimes \varepsilon_0 + \sum_{n\geq 1}^{\infty} \zeta_n \otimes \varepsilon_n$$

Then

$$\omega_{\widetilde{\eta}} \leqq \omega_{\widetilde{\xi}} \quad \text{on } M \otimes \mathbb{C}$$

Thus by Radon-Nykodim,

$\exists$ $$y' \in (M \otimes \mathbb{C})' = M' \otimes B(\ell^2)$$

s.t.

$$\widetilde{\eta} = y'\widetilde{\xi} \quad \& \quad \|y'\| \leqq 1.$$

Hence

$$\eta = (y'_{00} \ y'_{01} \cdots) \begin{bmatrix} \xi \\ \zeta_1 \\ \zeta_2 \\ \vdots \end{bmatrix}$$

$$= \underset{\substack{\| \\ : \\ x'}}{y'_{00}} \xi + \underset{\substack{\| \\ : \\ \zeta}}{\sum_{n\geq 1}^{\infty} y'_{0n} \xi_n}$$

Then $\|x'\| \leqq 1$, and

$$\|\zeta\| = \left\| (0 \ y'_{01} \ y'_{02} \cdots) \begin{bmatrix} 0 \\ \zeta_1 \\ \vdots \end{bmatrix} \right\|$$

$$\leqq \|(\qquad)\| \sqrt{\sum_{n\geq 1} \|\zeta_n\|^2}$$

$$\leqq 1 \cdot \|\varphi\|^{1/2}$$

(Claim)▨

By Claim,

we have $x_n' \in M'$ & $\zeta_n \in H$

s.t.

- $\| x_n' \| \leq 1$
- $\eta_n' = x_n' \xi_n + \zeta_n$
- $\| \zeta_n \| \leq \| \varphi_n^+ \|$ ⌟

But since

$$\lim_{n\to\omega} \|\zeta_n\| \leq \lim_{n\to\omega} \|\varphi_n^+\| = 0,$$

we have $(\zeta_n)_\omega = 0$, and

$$\eta' = (x_n' \xi_n + \zeta_n)_\omega$$

$$= (x_n')_\omega \xi + (\zeta_n)_\omega$$

$$= (x_n')_\omega \xi$$

$$=: a$$

▨

Thm. 4.5. (Raynaud) (H may not be std)

$$(\Pi^\omega M)' = \Pi^\omega M' \curvearrowright H_\omega$$ ⌟

proof

Let $a' \in (\Pi^\omega M)'$ & $\xi_1 \cdots \xi_m \in H_\omega$.

Naturally

$$(H \otimes \mathbb{C}^m)_\omega = H_\omega \otimes \mathbb{C}^m$$

$$\left(\sum_{i=1}^m \xi_n^i \otimes \varepsilon_i\right)_\omega \longleftrightarrow \sum_{i=1}^m (\xi_n^i)_\omega \otimes \varepsilon_i \in$$

Then

$$\Pi^\omega (M \otimes \underset{M_m(\mathbb{C})}{\cancel{\mathbb{C}\, 1_{\mathbb{C}^m}}}) = \Pi^\omega M \otimes \underset{M_m(\mathbb{C})}{\cancel{\mathbb{C}\, 1_{\mathbb{C}^m}}}$$

Thus

$$a' \otimes 1_{\mathbb{C}^m} \in \left(\Pi^\omega (M \otimes M_m(\mathbb{C}))\right)'$$

put $\xi := \sum_{i=1}^m \xi_i \otimes \varepsilon_i$

By Lem. 4.4, $^{\exists} a_{\otimes 1} \in \underset{\displaystyle (M' \otimes \mathbb{C} 1_{\mathbb{C}^m})}{\underset{\shortparallel}{\prod^{\omega} (M \otimes M_m(\mathbb{C}))'}}$

s.t.

$$(a' \otimes 1) \xi = (a \otimes 1) \xi .$$

i.e. $a' \xi_i = a \xi_i$ ▨.

We have completed our proof of Thm 4.3.

i.e. $(\prod^{\omega} M, H_\omega, J_\omega, P_\omega)$ is standard.

§4.2. Standard form of M^ω

Let $\varphi \in M_*^+$ faithful state.

Recall our isometry

$$\begin{tikzcd} w_\varphi : L^2(M^\omega, \varphi^\omega) \arrow[r] & (L^2(M,\varphi))_\omega \\ (x_n)^\omega \xi_{\varphi^\omega} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r, mapsto] & (x_n \xi_\varphi)_\omega \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

We put

$N := \prod^\omega M$

$H := L^2(M,\varphi)$ std.

$\xi_\omega := (\xi_\varphi)_\omega$

$p := p_\varphi \in N$ the proj from $L^2(M,\varphi)_\omega$ onto $[N' \xi_\omega]$.

$w := w_\varphi$

We will show

$$w w^* = p J_\omega p J_\omega ,$$

where $J_\omega = (J)_\omega$ defined in the prev §.

$$\begin{aligned} J_\omega p J_\omega H_\omega &= J_\omega p H_\omega \\ &= J_\omega \overline{N' \xi_\omega} \\ &= \overline{J_\omega N' J_\omega \xi_\omega} && \text{since } \xi_\omega \in P_\omega \ \& \ J_\omega \xi_\omega = J_\omega \\ &= \overline{N \xi_\omega} && \text{by Thm 4.5.} \end{aligned}$$

Hence

$$\begin{aligned} p J_\omega p J_\omega H_\omega &= p \overline{N \xi_\omega} \\ &= \overline{p N p \xi_\omega} . \end{aligned}$$

On the other hand,

$$w w^* H_\omega = \overline{\text{span}}\{ (x_n)_\omega \xi_\omega \mid (x_n)^\omega \in M^\omega \}$$

$$= \overline{\text{span}}\{ (x_n)_\omega p \xi_\omega \mid (x_n)^\omega \in M^\omega \}$$

$$= \overline{pNp\,\xi_\omega}$$

by Prop. 3.8.

Hence

$$w w^* = p J_\omega p J_\omega =: q$$

Then

$$w M^\omega w^* \overset{\text{Thm 3.1.}}{=} w (w^* \textstyle\prod^\omega M w) w^*$$

$$= q \textstyle\prod^\omega M \, q$$

$$\underset{\text{general fact.}}{\cong} p \textstyle\prod^\omega M \, p$$

general fact. ~~p : cyclic proj with cyclic vector in P_ω~~

Easy to show (proof of Thm 3.1 eq. (*))

$$w (x_n)^\omega w^* = (x_n)_\omega q.$$

Thus we have the following.

Thm. 4.6.

$\varphi \in M_*^+$ faithful state.

$p_\varphi : L^2(M,\varphi)_\omega \longrightarrow \overline{(\prod^\omega M)' \xi_\omega}$ proj

$q_\varphi := p_\varphi J_\omega p_\varphi J_\omega$

Then

(1) $w_\varphi w_\varphi^* = q_\varphi$

(2) $w_\varphi M^\omega w_\varphi^* = q_\varphi \prod^\omega M \, q_\varphi$

(3) $w_\varphi (x_n)^\omega w_\varphi^* = (x_n)_\omega q_\varphi$

for all $(x_n)^\omega \in M^\omega$

Cor 4.7.

$\varphi \in M_*^+$ as before.

Then $M^\omega \simeq q_\varphi \prod^\omega M q_\varphi$, and.

$(q_\varphi \prod^\omega M q_\varphi, q_\varphi H_\omega, q_\varphi J_\omega q_\varphi, q_\varphi P_\omega)$

is the standard form of M^ω ⌟

proof

Generally

For $p \in N$ proj, put $q := pJpJ$.

Then

$pNp \cong qNq$

&

(qNq, qH, qJq, qP) is std. ▨

Section 5 Modular automorphisms on M^ω

§5.1. Graph projection onto $G(S_\varphi)$

Let

$T : H \longrightarrow H$ closed linear

Hilb. sp./$\mathbb{C}$ or $\mathbb{R}$

$$G(T) := \{ (\xi, T\xi) \mid \xi \in D(T) \}$$

$$\subset H \oplus H$$

the graph of T, closed subsp.

The graph projection

$$P_T : H \oplus H \longrightarrow G(T)$$

is given by

$$P_T := \begin{bmatrix} (1+T^*T)^{-1} & T^*(1+TT^*)^{-1} \\ T(1+T^*T)^{-1} & TT^*(1+TT^*)^{-1} \end{bmatrix}$$

(Proof: Use $(1+TT^*)^{-1}T \subset \underset{\text{domain} = H}{T(1+T^*T)^{-1}}$)

☆ $T : H \longrightarrow H$ closed

Hilb/$\mathbb{C}$

$\rightsquigarrow T : H_{\mathbb{R}} \longrightarrow H_{\mathbb{R}}$ closed.

$D_{\mathbb{R}}(T) := D(T)$

$\langle \xi, \eta \rangle_{\mathbb{R}} := \mathrm{Re} \langle \xi, \eta \rangle$

since $\|\xi\|_{\mathbb{R}} = \|\xi\|$.

☆ $D_{\mathbb{R}}(T^*) = D(T^*)$ & $T^* \curvearrowright H_{\mathbb{R}} = T^* \curvearrowright H_{\mathbb{C}}$

Let $\eta \in D_{\mathbb{R}}(T^*)$

$$\Longleftrightarrow \ |\langle T\xi, \eta\rangle_{\mathbb{R}}| \le C\|\xi\| \quad \forall \xi \in D(T)$$

$$\overset{\displaystyle \|}{|\mathrm{Re}\langle T\xi, \eta\rangle|}$$

$$\Longleftrightarrow \ |\langle T\xi, \eta\rangle| \le C\|\xi\|$$

$$\Longleftrightarrow \ \eta \in D(T^*)$$

$\boxed{/}$

Note

$$H \oplus H = G(T) \oplus V G(T^*)$$

$$V := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$/\mathbb{C}$ or $/\mathbb{R}$.

Let

$S_\varphi : H_\varphi \longrightarrow H_\varphi$ the sharp operator

$F_\varphi : H_\varphi \longrightarrow H_\varphi$ the flat operator

Then by Tomita–Takesaki,

$$F_\varphi = S_\varphi^*,$$

which is still valid as a real operator.

Then

- $H \underset{\mathbb{R}}{\oplus} H = G(S_\varphi) \oplus_{\mathbb{R}} V\, G(F_\varphi)$

- $$P_{S_\varphi} = \begin{bmatrix} (1+\Delta_\varphi)^{-1} & J_\varphi \Delta_\varphi^{-\frac{1}{2}}(1+\Delta_\varphi^{-1})^{-1} \\ J_\varphi \Delta_\varphi^{\frac{1}{2}}(1+\Delta_\varphi)^{-1} & \Delta_\varphi(1+\Delta_\varphi)^{-1} \end{bmatrix}$$

§5.2 Main theorem.

Thm 5.1. (Kirchberg, Raynaud, AH)

M : σ-fin vN alg.

φ : faithful normal state on M

Then

$$\sigma_t^{\varphi^\omega} = (\sigma_t^\varphi)^\omega \quad \text{for all } t \in \mathbb{R}$$

i.e.

$$\sigma_t^{\varphi^\omega}\left((x_n)^\omega\right) = \left(\sigma_t^\varphi(x_n)\right)^\omega$$

for all $(x_n)^\omega \in M^\omega$

$\lrcorner$

Proof of Thm 5.1.

Recall

$$\begin{tikzcd} \mathcal{W} : L^2(M^\omega, \varphi^\omega) \arrow[r] & L^2(M, \varphi)_\omega \\ (x_n)^\omega \xi_{\varphi^\omega} \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (x_n \xi_\varphi)_\omega \arrow[u, phantom, sloped, "\in"] \\ & (x_n)_\omega \xi_\omega \arrow[u, phantom, sloped, "="] \end{tikzcd}$$

NOTATIONS

$N := \prod^\omega M$

$H := L^2(M, \varphi)$

$\xi_\omega := (\xi_\varphi)_\omega$

$p :=$ the proj H_ω onto $\overline{N' \xi_{\omega}}$

$q := pJpJ$.

Then we have.

$$\begin{cases} w w^* = q \\ w M^\omega w^* = q \Pi^\omega M q \end{cases}$$

by Thm. 4.6.

Let

$$\begin{array}{cccc} \theta : & M^\omega & \longrightarrow & q \Pi^\omega M q \\ & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ & x & \longmapsto & w x w^* . \end{array}$$

Then

$$\theta((x_n)^\omega) = \underset{\text{Thm 4.6 (3)}}{(x_n)_\omega q}$$

$$\begin{array}{ccc} qNq & \curvearrowright & qH_\omega \\ & & \rotatebox[origin=c]{90}{$\in$} \\ & & \xi_\omega \end{array}$$ cyclic & separating vector.

Set

$\psi \in (qNq)_*$ faithful state.

$$\psi(x) := \langle x \xi_\omega , \xi_\omega \rangle \quad \text{for } x \in qNq .$$

Then $\psi \circ \theta = \varphi^\omega$.

Since

$$\begin{aligned} \psi \circ \theta((x_n)^\omega) &= \psi((x_n)_\omega q) \\ &= \langle (x_n)_\omega \xi_\omega , \xi_\omega \rangle \\ &= \varphi^\omega((x_n)^\omega) . \end{aligned}$$

Hence we have.

$$\begin{tikzcd}
\sigma_t^{\psi} \circ \theta = \theta \circ \sigma_t^{\varphi^\omega} \quad \text{on } M^\omega . \\
\Delta_\psi \arrow[u, squiggly]
\end{tikzcd}$$

need to compare

$$\Delta_\psi^{\frac{1}{2}} \quad \text{with} \quad \text{"}(\Delta_\varphi^{\frac{1}{2}})_\omega\text{"}$$

Idea

Use graphs:

$$\begin{tikzcd}
G(S_\psi) \quad \& \quad G(S_\varphi)_\omega \\
\text{ultraproduct Hilbert sp.} \arrow[u]
\end{tikzcd}$$

First of all,

$$G(S_\psi) \oplus_{\mathbb{R}} V G(F_\psi) = qH_\omega \oplus_{\mathbb{R}} qH_\omega \quad — ①$$

omega.

Next,

$$\begin{tikzcd}
G(S_\varphi) \oplus_{\mathbb{R}} V G(F_\varphi) = H \oplus H \arrow[d, "\text{ultraproduct of ``real'' Hilb. sp.}"] \\
G(S_\psi)_\omega \oplus_{\mathbb{R}} V G(F_\varphi)_\omega = H_\omega \oplus_{\mathbb{R}} H_\omega \quad — ②
\end{tikzcd}$$

Now

$$\theta(M^\omega)\xi_\omega = qNq\,\xi_\omega \overset{\text{core}}{\subset} D(S_\psi)$$

$$qN'q\,\xi_\omega \overset{\text{core}}{\subset} D(F_\psi)$$

$$\begin{aligned} S_4\, \theta((x_n)^\omega)\xi_\omega &= \theta((x_n^*)^\omega)\xi_\omega \\ &= (x_n^*)_\omega \xi_\omega \\ &= (x_n^* \xi_\varphi)_\omega \\ &= (S_\varphi x_n \xi_\varphi)_\omega , \end{aligned}$$

which implies

$$G(S_4) \subset G(S_\varphi)_\omega \cap (qH_\omega \underset{\mathbb{R}}{\oplus} qH_\omega)$$

$\longrightarrow$ ③

On F_4,

$$\Pi^\omega M' = (\Pi^\omega M)' = N'$$

Let us replace M with M' in our discussion.

Then for $(x_n') \in \mathcal{M}^\omega(M')$,

$(x_n')_\omega$ commutes with

$$p' : H_\omega \longrightarrow [(\Pi^\omega M')'\xi_\omega] \overset{\text{Thm 4.5}}{=} [\Pi^\omega M \xi_\omega]$$

$$p' = J_\omega p J_\omega \qquad \text{by Prop 3.8 (1).}$$

$\rightsquigarrow$ $q' := p' J_\omega p' J_\omega = J_\omega p J_\omega p = q.$

And,

$$\begin{tikzcd} \theta' : (M')^\omega \arrow[r, "\sim"] & qN'q \\ (x_n')^\omega \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (x_n')_\omega q \arrow[u, phantom, sloped, "\ni"] \end{tikzcd}$$

by Thm 4.6 (3).

Thus.

$$\begin{aligned} F_4\, \theta'((x_n')^\omega)\xi_\omega &= \theta'((x_n'^*)^\omega)\xi_\omega \\ &= (x_n'^* \xi_\varphi)_\omega \\ &= (F_4 x_n' \xi_\varphi)_\omega \end{aligned}$$

Therefore

$$G(F_\psi) \subset G(F_\varphi)_\omega \cap (q H_\omega \oplus_{\mathbb{R}} q H_\omega),$$

and

$$V G(F_\psi) \subset V\, G(F_\varphi)_\omega \cap (q H_\omega \oplus_{\mathbb{R}} q H_\omega)$$

— ④

We will express this situation

in terms of $\overbrace{\text{real}}$ projections:

$E : H_\omega \oplus_{\mathbb{R}} H_\omega \longrightarrow G(S_\varphi)_\omega$

$F : H_\omega \oplus_{\mathbb{R}} H_\omega \longrightarrow q H_\omega \oplus_{\mathbb{R}} q H_\omega$

$P : H_\omega \oplus_{\mathbb{R}} H_\omega \longrightarrow G(S_\psi)$

$Q : H_\omega \oplus_{\mathbb{R}} H_\omega \longrightarrow V G(F_\psi)$

Then

① $\rightsquigarrow$ $P + Q = F$

② $\rightsquigarrow$ $E^\perp : H_\omega \oplus_{\mathbb{R}} H_\omega \longrightarrow V G(F_\varphi)_\omega$

③ $\rightsquigarrow$ $P \leqq E \wedge F$

④ $\rightsquigarrow$ $Q \leq E^\perp \wedge F$

<u>Claim 1.</u>

E & F commute & $P = EF$,

$Q = E^\perp F$

Let us prove the above afterwards.

Recall the graph proj. of S_φ in §5.1.

Then

$$E = \left(\begin{bmatrix} \overset{\overset{a}{\shortparallel}}{(1+\Delta_\varphi)^{-1}} & \overset{\overset{b}{\shortparallel}}{J_\varphi \Delta_\varphi^{-\frac{1}{2}} (1+\Delta_\varphi^{-1})^{-1}} \\ \underset{\underset{c}{\shortparallel}}{J_\varphi \Delta_\varphi^{\frac{1}{2}} (1+\Delta_\varphi)^{-1}} & \underset{\underset{d}{\shortparallel}}{\Delta_\varphi^{-1} (1+\Delta_\varphi^{-1})^{-1}} \end{bmatrix} \right)_\omega$$

$$= \begin{bmatrix} a_\omega & b_\omega \\ c_\omega & d_\omega \end{bmatrix} \curvearrowright H_\omega \oplus_{\mathbb{R}} H_\omega$$

$$P = \begin{bmatrix} (1+\Delta_\psi)^{-1} & J_\psi \Delta_\psi^{-\frac{1}{2}} (1+\Delta_\psi^{-1})^{-1} \\ J_\psi \Delta_\psi^{\frac{1}{2}} (1+\Delta_\psi)^{-1} & \Delta_\psi^{-1} (1+\Delta_\psi^{-1})^{-1} \end{bmatrix}.$$

$$F = \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

By Claim 1,

$$q a_\omega = a_\omega q = (\underset{\uparrow}{\boxed{1}} + \Delta_\psi)^{-1}$$

Of course this 1 means q,

That is,

$$(1+\Delta_\psi)^{-1} = ((1+\Delta_\varphi)^{-1})_\omega q$$

<u>Claim 2.</u>

$$\Delta_\psi^{it} = (\Delta_\varphi^{it})_\omega q$$

⌟

<u>Proof of Thm 5.1</u>

Then for $x = (x_n)^\omega \in M^\omega$,

$$\theta(\sigma_t^{\varphi^\omega}(x)) = \sigma_t^\psi(\theta(x))$$
$$= \Delta_\psi^{it} (x_n)_\omega q \Delta_\psi^{-it}$$
$$= (\Delta_\varphi^{it} x_n \Delta_\varphi^{-it})_\omega q$$

$$= \theta((\sigma_t^{\varphi}(x_n))^{\omega})$$

▨

We will prove Claim 1 & 2 in what follows.

<u>proof of Claim</u> 1

$$\begin{array}{ccc} E\wedge F + E^{\perp}\wedge F & \leqq F \\ \mathbin{\rotatebox{90}{$\leqq$}} & \\ P + Q & = F \end{array}$$

Hence $P = E\wedge F$, $Q = E^{\perp}\wedge F$

$$E\wedge F + E^{\perp}\wedge F = F.$$

Note

$$E\wedge F \leqq FEE$$
$$E^{\perp}\wedge F \leqq FE^{\perp}F$$

addition

& $FEF + FE^{\perp}F = F$

| $\sim$ $E\wedge F = FEF$ ①, $E^{\perp}\wedge F = FE^{\perp}F$

Then $(EFE)^2 = E\underbrace{FE}_{E\wedge F}FE$

$= E\wedge F$ projection.

Since $EFE \geqq 0$, EFE itself proj.

& $EFE = E\wedge F$ ②

Then

$$(EF-FE)^*(EF-FE)$$

$$= \underbrace{FEF}_{\substack{E\wedge F\\ ①}} - \underbrace{FEFE}_{\substack{E\wedge F\\ ②}} - \underbrace{EFEF}_{\substack{E\wedge F\\ ②}} + \underbrace{EFE}_{\substack{E\wedge F\\ ②}}$$

$= 0$

▨

On Claim 2.

We have.

$$(1+\Delta_\psi)^{-1} = ((1+\Delta_\varphi)^{-1})_\omega q$$

Let
$$a := (1+\Delta_\varphi)^{-1}$$

Then

- $0 \leqq a \leqq 1$
- $Sp_p(a) \not\ni 0, 1 \Leftrightarrow Sp(\Delta_\varphi) \not\ni 0.$
  point spectrum.
- $Sp_p(a_\omega q) \not\ni 0, 1 \Leftrightarrow Sp(\Delta_\psi) \not\ni 0.$

Claim 2 follows from the follong Lema.

by putting $g(x) := (x^{-1}-1)^{it}$.

Lem 5.2.

Let $g : [0,1] \to \mathbb{C}$ function, bounded

s.t.
$g|_{(0,1)}$ is continuous.

Then

(1) $(g(a))_\omega$ commutes with q

(2) $(g(a))_\omega q = g(a_\omega q)$ $\lrcorner$

Proof

First, note that

if $h \in B(H)_{sa}$ $0 \leqq h \leqq 1$
satisfies $0, 1 \notin Sp_p(h)$,

then $g(h)$ is not depending on
$g(0)$ & $g(1)$

because

$$g(t) = \underset{\underset{0}{\downarrow}}{g(0) 1_{\{0\}}(t)} + g(t) 1_{(0,1)}(t) + \underset{\underset{0}{\downarrow}}{g(1) 1_{\{1\}}(t)}$$

For a polynomial f.

$$(f(a))_\omega = f(a_\omega).$$

Then Weierstraß thm states

$$(f(a))_\omega = (f(a_\omega)) \quad — (*)$$

for all $f \in C[0,1]$.

Let

$$K := q H_\omega \circlearrowleft a_\omega \text{ inv.}$$

Since $0, 1 \notin Sp_p(a_\omega q)$.

$$K = 1_{(0,1)}(a_\omega) K$$

$$= \bigvee_{0<\varepsilon<\frac{1}{2}} K_\varepsilon,$$

where

$$K_\varepsilon := 1_{(\varepsilon, 1-\varepsilon)}(a_\omega) K.$$

Want ~~to show~~

$$(g(a))_\omega = g(a_\omega) \quad \text{on } K_\varepsilon.$$

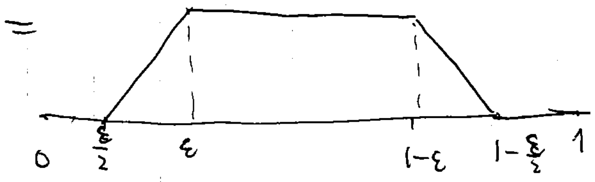
for all ε.

Fix ε.

Take $h \in C[0,1]$ so that

$$h = g \text{ on } [\tfrac{\varepsilon}{2}, 1-\tfrac{\varepsilon}{2}].$$

Let f_ε =



Then for $\xi \in K_\varepsilon$,

$$g(a_\omega)\xi = g(a_\omega) 1_{(\varepsilon, 1-\varepsilon)}(a_\omega)\xi$$
$$= h(a_\omega) 1_{(\varepsilon, 1-\varepsilon)}(a_\omega)\xi$$
$$= (h(a))_\omega \xi \quad — ①.$$

$$f_\varepsilon(a_\omega)\xi = f_\varepsilon(a_\omega) 1_{(\varepsilon, 1-\varepsilon)}(a_\omega)\xi$$
$$= 1_{(\varepsilon, 1-\varepsilon)}(a_\omega)\xi$$
$$= \xi.$$

Thus

$$(h(a))_\omega \xi = (h(a))_\omega f_\varepsilon(a_\omega)\xi$$
$$= (h(a))_\omega (f_\varepsilon(a))_\omega \xi$$
$$= (h(a) f_\varepsilon(a))_\omega \xi$$
$$= (g(a) f_\varepsilon(a))_\omega \xi$$
$$= (g(a))_\omega (f_\varepsilon(a))_\omega \xi$$
$$= (g(a))_\omega \xi \quad — ②$$

From ① & ②, we are done ∎

Cor 5.3. (Spectral characterization of M^ω)

TFAE

(1) $(x_n) \in M^\omega$

(2) $\forall \varepsilon > 0.\ \exists a > 0.\ (y_n) \in M^\omega$

s.t.

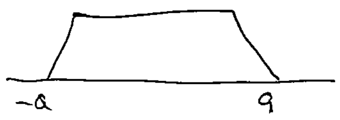
(1) $\lim_{n \to \omega} \| x_n - y_n \|_\varphi^\# < \varepsilon$

(2) $y_n \in M(\sigma^\varphi, [-a, a])\ \forall n.$

(3) $(x_n) \in \ell^\infty$ is σ^φ-equicont.

See Masuda-Tomatsu Rohlin flow & Classification of discrete Kac ...

Proof Apply Lem 5.2 to g with

$\hat{g} =$ 

See [AH]. ∎

Section 6, Ueda's problem

$$\mathcal{M}_\omega := \{ (x_n) \in \ell^\infty \mid \lim_{n\to\omega} \| x_n\varphi - \varphi x_n \| = 0 \quad \forall \varphi \in M_* \}$$

unital C^*-alg

Then

$$\underset{\substack{\uparrow \\ \text{Cauchy-Schwarz}}}{\mathcal{I}_\omega \subset} \mathcal{M}_\omega \underset{\uparrow}{\subset} \mathcal{M}^\omega$$

If $(y_n) \in \mathcal{I}_\omega$

$$\| y_n x_n \|_\varphi^2 = \varphi(x_n^* y_n^* y_n x_n)$$

$$= x_n\varphi(x_n^* y_n^* y_n)$$

$$= \underset{\swarrow 0}{[x_n, \varphi]} (\underset{\text{bdd}}{\boxed{x_n^* y_n^*}} y_n) \uparrow$$

$$+ \varphi(\boxed{x_n x_n^* y_n^*} y_n)$$

bdd

Cauchy-Schwarz.

Defn. 6.1 (Connes)

$$M_\omega := \mathcal{M}_\omega / \mathcal{I}_\omega$$

the asymptotic centralizer of M.

☆ Useful characterization. (Araki-Powers-Størmer)

$$(x_n) \in \mathcal{M}_\omega \Leftrightarrow \lim_{n\to\omega} \| x_n\xi - \xi x_n \| = 0 \quad \forall \xi \in H.$$

where

$$\xi a := J a^* J \xi.$$

☆ M_ω is a W^*-alg. follows from Thm ~~6.4~~ 6.5

☆ $M_\omega \subset M^\omega$.

$$\cup$$

$$M = \{ (x)^\omega \mid x \in M \}$$

constant sequence.

Lem 6.2

$M_\omega \subset M' \cap M^\omega$ ⌟

proof

$(x_n)^\omega \in M_\omega$. $y \in M$.

$x_n y - y x_n \xrightarrow[n \to \omega]{s^*} 0$?

$$(x_n y - y x_n)\xi = x_n y \xi - y x_n \xi$$
$$\sim (y\xi) x_n - y \xi x_n = 0$$

$\square$

✲ $Z(M) \subset M_\omega$.

Defn. 6.3

M full factor

$\Leftrightarrow M_\omega = \mathbb{C}$. ⌟

✲ 3 types of factors.

$$M_\omega \begin{cases} \mathbb{C} \\ \text{abelian} \neq \mathbb{C} \to \infty\text{-dim.} \\ \text{non-abelian} \to \text{McDuff} \\ \qquad \text{i.e. } M \simeq M \otimes R_0 \end{cases}$$

Problem (Ueda)

M full $\Leftrightarrow M' \cap M^\omega = \mathbb{C}$

$\Leftarrow$ trivial.

~~Lem~~

Idea. Show

$$M_\omega = (M' \cap M^\omega)_{\varphi^\omega}$$

$$\left[\begin{aligned} M_\varphi &:= \{x \mid x\varphi = \varphi x\} \\ &= \{x \mid \sigma_t^\varphi(x) \underset{\forall t}{=} x\} \end{aligned}\right]$$

centralizer

$\subset$ is trivial.

Indeed $(x_n)^\omega \in M_\omega$.

$(y_n)^\omega \in M^\omega$.

$$\begin{aligned} ((x_n)^\omega \varphi^\omega)((y_n)^\omega) &= \varphi^\omega((y_n x_n)^\omega) \\ &= \lim_{n\to\omega} \varphi(y_n x_n) \\ &= \lim_{n\to\omega} \varphi(x_n y_n) \\ &= (\varphi^\omega (x_n)^\omega)((y_n)^\omega) \end{aligned}$$

Lem. 6.4

For $(x_n), (y_n) \in \mathcal{M}^\omega$,

$$\|(x_n)^\omega \varphi^\omega - \varphi^\omega (y_n)^\omega\|_{M_*^\omega} = \lim_{n\to\omega} \|x_n\varphi - \varphi x_n\|_{M_*}$$

Thm. 6.~~4~~5

$$M_\omega = (M' \cap M^\omega)_{\varphi^\omega}$$

proof

$\subset$ ok

$\supset$ $(x_n)^\omega \in (M' \cap M^\omega)_{\varphi^\omega}$.

By Lem,

$$\lim_{n\to\omega} \|x_n\varphi - \varphi x_n\| = 0.$$

Then for $y \in M$

$$\|\underset{\sim}{x_n} y\varphi - y\varphi x_n\|$$

$$\sim \|y x_n \varphi - y x_n \varphi\| = 0$$

& $\{y\varphi \mid y \in M\} \subset M_*$ normdense

5b **proof of Lem.**

$C_1 := \| (x_n)^\omega \varphi^\omega - \varphi^\omega (y_n)^\omega \|$

$C_2 := \lim_{n \to \omega} \| x_n \varphi - \varphi y_n \|$

Then $\forall \varepsilon > 0$ $\exists$ $a \in M^\omega$ $\|a\| \lneq 1$

s.t.

$$C_1 - \varepsilon < | ((x_n)^\omega \varphi^\omega - \varphi^\omega (y_n)^\omega)(a) |$$

~~Let~~ Since $M^\omega = \mathcal{M}^\omega / \mathcal{I}^\omega$,

$\exists$ $(a_n) \in \mathcal{M}^\omega$ s.t. $\|(a_n)\| = \sup_n \|a_n\| < 1$

s.t. $(a_n)^\omega = a$

Then

$$C_1 - \varepsilon < \lim_{n \to \omega} | (x_n \varphi - \varphi y_n)(a_n) |$$

$$\leq C_2 .$$

Thus $C_1 \leq C_2$

Take $a_n \in M$ $\|a_n\| \leq 1$.

$$\| x_n \varphi - \varphi y_n \| - \frac{1}{n} \leq |(x_n \varphi - \varphi y_n)(a_n).|$$

By Prop , $\exists (b_n) \in \mathcal{M}^\omega$, $(c_n) \in \mathcal{L}_\omega$, $(d_n) \in \mathcal{L}_\omega^*$

s.t.

$$a_n = b_n + c_n + d_n$$

$$\| (b_n)^\omega \| \leq \lim_{n \to \omega} \| a_n \| \leq 1$$

Then

$$(x_n \varphi - \varphi y_n)(a_n)$$

$$= (x_n \varphi - \varphi y_n)(b_n)$$

$$+ \underbrace{(x_n \varphi - \underline{\varphi y_n})(c_n)}_{\to 0} \quad (c_n) \in \mathcal{L}_\omega$$

$$+ (\underbrace{x_n \varphi}_{\to 0} - \varphi y_n)(d_n) \quad (d_n) \in \mathcal{L}_\omega^*$$

$x_n \varphi (c_n) = \varphi(c_n x_n)$

$\downarrow$ $(c_n x_n) \in \mathcal{I}_\omega \mathcal{M}_\omega \subset \mathcal{L}_\omega$

0

$\varphi y_n (d_n) \longrightarrow 0$

Hence

$$\begin{array}{ccc}
\lim\limits_{n\to\omega} |(x_n\varphi - \varphi y_n)(a_n)| & = & \lim\limits_{n\to\omega} |(x_n\varphi - \varphi y_n)(b_n)| \\
\rotatebox{-90}{$\geq$} & & \rotatebox{90}{$\geq$} \\
\lim\limits_{n\to\omega} \|x_n\varphi - \varphi y_n\| & & \lim\limits_{n\to\omega} \|x_n\varphi - \varphi y_n\| \, \|b_n\| \\
\| & & \rotatebox{90}{$\geq$} \\
C_2 & & C_1
\end{array}$$

Hence $C_1 \geq C_2$

$\boxed{}$

So, If M full,

$$(M' \cap M^\omega)_{\varphi^\omega} = \mathbb{C}.$$

Recall the general fact:

$$N_\psi = \mathbb{C} \implies N = \mathbb{C} \text{ or type III}_1 \text{ factor.}$$

Therefore

$$M' \cap M^\omega = \mathbb{C} \quad \text{or} \quad \text{type III}_1 \text{ factor.}$$

want to show $\downarrow$

M non-full

i.e.

$\exists$ non-trivial

ω-central seq.

Lem. 6.6

Let N III_1 factor. $\varphi \in N_*^+$ faithful state

$0 < \lambda < 1$.

$\forall \varepsilon > 0$. $\exists \{f_{ij}\}_{i,j=1}^{2}$: matrix units.

s.t.

$$\left.\begin{aligned} &\| f_{ii} \varphi - \varphi f_{ii} \| < \varepsilon \\ &\| f_{12} \varphi - \lambda^{-1} \varphi f_{12} \| < \varepsilon \\ &\| f_{21} \varphi - \lambda \varphi f_{21} \| < \varepsilon \end{aligned}\right\} \text{—} (*)$$

$\lrcorner$

Sketch of proof

$M_2 \hookrightarrow N$.

$N \underset{\alpha}{\xrightarrow{\sim}} M_2(\mathbb{C}) \otimes (M_2' \cap N)$

$$\psi(x) := \Big(\underset{\substack{\uparrow \\ \text{powers state}}}{\mathrm{Tr}\begin{bmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{bmatrix}} \otimes \underset{\substack{\uparrow \\ \text{some state}}}{\chi} \Big)(x)$$

Connes – Størmer transitivity.

$\exists u \in \mathcal{U}(N)$

$\varphi \underset{\varepsilon}{\sim} u \psi u^*$.

$f_{ij} := u\, \alpha^{-1}(e_{ij} \otimes 1)\, u^*$. $\leftarrow$ rotate $M_2(\mathbb{C})$.

▨.

Now suppose $\underset{\underset{N}{\shortparallel}}{M' \cap M^\omega}$ III_1 factor.

$0 < \lambda < 1$ fixed.

By Lemma 6.6,

$\forall \varepsilon > 0$. $\exists f_{ij} \in N$ matrix units

s.t. $(*)$ holds. for φ^ω

f_{ii} almost commute φ^ω.

$$\varphi^\omega(f_{11}) = \varphi^\omega(f_{12} f_{21}) = f_{21} \varphi^\omega(f_{12}) \sim \lambda \underset{\underset{\lambda \varphi^\omega(f_{22})}{\shortparallel}}{\varphi^\omega(f_{21} f_{12})}$$

Want to construct a ω-central seq.
from f_{11} & f_{22}.

$\varepsilon = \frac{1}{n}$. $f_{ij}^{(n)} \in M' \cap M^\omega = N$

$\| f_{ij}^{(n)} \varphi^\omega \underset{\lambda^{i-j}}{\wedge} - \varphi^\omega f_{ii}^{(n)} \| < \frac{1}{n}$

$\|$

$\lim_{k \to \omega} \| f_{ij}^{(n)}(k) \varphi - \lambda^{i-j} \varphi f_{ij}^{(n)}(k) \|$

$\uparrow$ matrix units for each k. .

$f_{ij}^{(1)}(1)$ - - -

$f_{ij}^{(2)}(1)$ $\searrow$ diagonal. argumet.

$\vdots$

(index selection trick)

We have.

$f_{ij} \in M' \cap M^\omega$ matrix u.t.

s.t.

$f_{ij} \varphi^\omega = \lambda^{i-j} \varphi^\omega f_{ij}$

Then $f_{ii} \in (M' \cap M^\omega)_{\varphi^\omega} = \mathbb{C}$.

$f_{ii} = 1$ or 0 ↯

But $\varphi^\omega(f_{11}) = \lambda^{\cancel{\ }} \varphi^\omega(f_{22})$ Pr~~oof~~ ∎

<u>Thm.</u> 6.7

$M_\omega = \mathbb{C} \iff M' \cap M^\omega = \mathbb{C}$

<u>Cor.</u> 6.8

M full $\Rightarrow$ M^ω factor.

Section 7 M^ω is a factor?

Thm. 7.1.

- M factor not of ~~no~~ type III_0

  $\Rightarrow M^\omega$ factor

- M type III_0 factor

  $\Rightarrow M^\omega$ non-factor ┘

~~Proof~~

M type I

$B(H)^\omega = B(H)$

type II_1

M^ω II_1 factor

II_∞

$(M \otimes B(H))^\omega = M^\omega \otimes B(H)$

II_∞-factor.

$\mathrm{III}_\lambda \Rightarrow ?$

Idea: Use $d(M)$

diameter of M. ┘

$S_n(M) := \{\varphi \in M_*^+ \mid \varphi(1) = 1\}$

$\varphi \sim \psi \Leftrightarrow \varphi = \lim_{n\to\infty} u_n \psi u_n^* \quad \exists u_n \in M^u.$

$$d([\varphi],[\psi]) := \inf_{u \in \mathcal{U}(M)} \|\varphi - u\psi u^*\| \quad \leqq 2$$

$S_n(M)/\sim$ metric d.

Defn. 7.2

$$d(M) := \sup_{\varphi, \psi \in S_n(M)} d([\varphi],[\psi])$$

M not factor $\Rightarrow d(M) = 2$

Thm. 7.3 M factor

$$\therefore \quad d(M) = \begin{cases} 2\left(1-\frac{1}{n}\right) & M \text{ type } \mathrm{I}_n \\ 2 & \mathrm{II} \\ 2\,\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}} & \mathrm{III}_\lambda \\ & 0 \leq \lambda \leq 1 \end{cases}$$

$\lrcorner$

Let us recall $\Pi^\omega M \curvearrowright H_\omega$ standard.

Thm. 7.4

$$d(\Pi^\omega M) = d(M)$$

$\lrcorner$

proof

Let. $\varphi, \psi \in (\Pi^\omega M)_*^+$ states.

$\exists! \xi, \eta \in P_\omega$

s.t. $\varphi = \omega_{\xi}$, $\psi = \omega_{\eta}$

$\xi = (\xi_n)_\omega$, ~~$\xi_\psi = (\xi$~~

$\eta = (\eta_n)_\omega$.

$(\xi_n, \eta_n \in P)$

Let $\varphi_n := \omega_{\xi_n}$, $\psi_n := \omega_{\eta_n}$

$\exists\ u_n \in \mathcal{U}(M)$

s.t.

$$\|\varphi_n - u_n \psi_n u_n^*\| < d([\varphi_n], [\psi_n]) + \frac{1}{n}.$$

Then $\underset{u}{\overset{\shortparallel}{(u_n)_\omega}} \in \Pi^\omega M$.

$$\|\varphi - u\psi u^*\| = \lim_{n\to\omega} \|\varphi_n - u_n \psi u_n^*\|$$

$$\leq \lim_{n\to\omega} d([\varphi_n], [\psi_n])$$

$$\leq d(M).$$

$$\therefore \quad d(\Pi^\omega M) \leq d(M)$$

Take $\varphi_n, \psi_n \in S_n(M)$

$u_n \in \mathcal{U}(M)$

s.t.

$$\|\varphi_n - u_n \psi_n u_n^*\| \geqq d(M) - \frac{1}{n}$$

Then

$$\lim_{n \to \omega} \|\varphi_n - u_n \psi_n u_n^*\| \geqq d(M)$$

$\|$ ← because we're treating $\prod^\omega M$

$$\|\varphi - u \psi u^*\|$$

$\wedge\!\!/$

$$d(\prod^\omega M),$$

where $\varphi = (\varphi_n)_\omega$, $\psi = (\psi_n)_\omega$ ∎

Cor. 7.5.

$0 \lneqq \lambda \leqq 1$

M type III_λ factor

$\Rightarrow \prod^\omega M$, M^ω III_λ factors ⌟

$\overset{\uparrow}{\text{corner of } \prod^\omega M}$

proof

$$d(\prod^\omega M) = 2\frac{1-\sqrt{\lambda}}{1+\sqrt{\lambda}} \lneqq 2 \to \prod^\omega M \quad \mathrm{III}_\lambda$$

∎

- **Type III_0 case**

Let

$$M = M_\varphi \rtimes_\theta \mathbb{Z}$$

be the discrete decomposition. wrt.

φ lacunary wt.

Let U be the implementing unitary of θ.

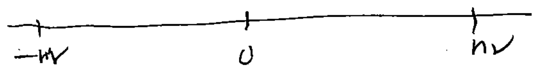
$\exists$ $\nu > 0$ s.t. $U \in M(\sigma^\varphi, [\nu, \infty))$

In particular

$$M_\varphi U^n \in M(\sigma^\varphi, [n\nu, \infty)) \leftarrow \text{If } n \geq 0$$

or $[-\infty, n\nu] \leftarrow$ If $n \leq 0$.

→ going for.



Let $(x_n) \in \mathcal{M}^\omega$ &

$$x_n = \sum_{k \in \mathbb{Z}} x_n(k) U^k$$

the Fourier series of x_n.

Since (x_n) is a multiplier seq.,

by Cor 5.3, the spectrum of x_n is

almost contained in $[-a, a]$.

for any n. (or big n).

This means. for big k, $x_n(k) \underset{\|\cdot\|_\varphi^\#}{\sim} 0$.

Therefore x_n is "almost" contained

in the algebraic span of $M_\varphi^\omega U^k$

This implies

$$M^\omega = M_\varphi^\omega \rtimes_\theta \mathbb{Z}.$$

$$\cup$$

$$Z(M^\omega) \supset (Z(M_\varphi)^\omega)^{\theta^\omega}.$$

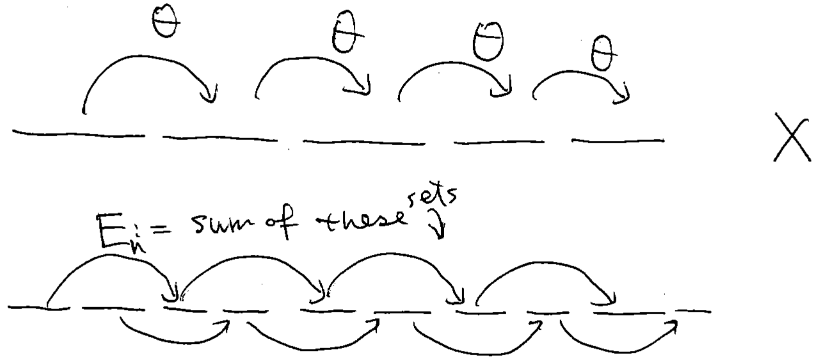
$$\uparrow$$

We will show is non-trivial.

$\theta \curvearrowright Z(M_\varphi) = L^\infty(X, \mu)$ ← prob. measure.

ergodic & non-singular.

Thanks to Rochlin thm.



Then · $\mu(E_n) \underset{\frac{1}{n}}{\sim} \frac{1}{2}$

· $\mu(E_n \Delta \theta(E_n)) < \frac{1}{n}$

Then $e_n := 1_{E_n} \in Z(M_\varphi)$

$$e := (e_n) \in \left(Z(M_\varphi)^\omega\right)^{\theta^\omega}$$

But $\mu(e) = \frac{1}{2}$. non-triv.

▨

Note that

$$E_M : \begin{array}{ccc} M^\omega & \longrightarrow & \overset{\swarrow \text{Type III}_0}{M} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (x_n)^\omega & \longmapsto & \lim\limits_{n\to\omega} x_n \end{array}$$

is a faithful normal cond. exp.

By Tomiyama's thm, M^ω is of Type $\underline{\text{III}}$.