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§1.1 Tomita - Takesaki theory

M ; vN alg

$\varphi \in W(M)$. (weight a defn ?)

$\rightsquigarrow$ GNS repn $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$

Hilb sp. repn.

$\Lambda_\varphi : n_\varphi \to H_\varphi$. canonical injection

$\langle \Lambda_\varphi(x), \Lambda_\varphi(y) \rangle := \varphi(y^* x)$.

$\pi_\varphi(x) \Lambda_\varphi(y) := \Lambda_\varphi(xy)$

Let

$S_\varphi^0 : H_\varphi \longrightarrow H_\varphi$ map.

$D(S_\varphi^0) := \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$

$S_\varphi^0 \Lambda_\varphi(x) = \Lambda_\varphi(x^*)$

$\rightsquigarrow$ S_φ^0 is
- densely defined
- closable
- conj - linear operator.

$S_\varphi := \overline{S_\varphi^0}$ closure $\curvearrowright H_\varphi$

$= J_\varphi \Delta_\varphi^{\frac{1}{2}}$ polar decomp.

- conj - linear
- $J_\varphi^2 = 1_{H_\varphi}$
- $\langle J_\varphi \xi, J_\varphi \eta \rangle = \langle \eta, \xi \rangle$

positive non-singular operator.

Thm. 1.1 (Tomita)

- $\Delta_\varphi^{it} \pi_\varphi(M) \Delta_\varphi^{-it} = \pi_\varphi(M)$, $\forall t \in \mathbb{R}$
- $J_\varphi \pi_\varphi(M) J_\varphi = \pi_\varphi(M)'$

For $t \in \mathbb{R}$, define $\sigma_t^\varphi \in \mathrm{Aut}(M)$ by

$$\pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it}\, \pi_\varphi(x)\, \Delta_\varphi^{-it}.$$

$\uparrow$ modular auto.

Thm 1.2 (KMS condition)

- $\varphi \circ \sigma_t^\varphi = \varphi$, $\forall t \in \mathbb{R}$
- $\forall x, y \in n_\varphi \cap n_\varphi^*$,

  $\exists F(z)$, hol on D

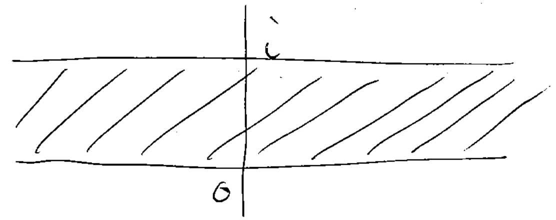
  bdd cont on $\overline{D}$

  s.t.

  $F(t) = \varphi(\sigma_t^\varphi(x) y)$

  $F(t+i) = \varphi(y\, \sigma_t^\varphi(x))$

trace条件の代わり

$D =$  open.

☆ If $\alpha_t \in \mathrm{Aut}(M)$ one-parameter group

(i.e. $\alpha : \mathbb{R} \longrightarrow \mathrm{Aut}(M)$ "continuous")

u-topology

$\alpha_{s+t} = \alpha_s \alpha_t$

satisfies the KMS cond. wrt φ.

then $\alpha_t = \sigma_t^\varphi$

Cor. 1.3 (centralizer)

Put $M_\varphi := \{ x \in M \mid "x\varphi = \varphi x" \}$

Then $M_\varphi = M^{\sigma^\varphi}$ $\big(:= \{ x \in M \mid \sigma_t^\varphi(x) = x \ \forall t \}\big)$

fixed pt alg.

where "$x\varphi = \varphi x$" means.

$$\begin{cases} \cdot\ x\, m_\varphi \subset m_\varphi \ \&\ m_\varphi\, x \subset m_\varphi \\ \cdot\ \varphi(ax) = \varphi(xa) \quad \forall a \in m_\varphi. \end{cases}$$

§1.2 Radon–Nykodim cocycle

For $\varphi, \psi \in W(M)$,

$\exists !$ ~~$D\varphi : D$~~ $u_t \in M^u$, $t \in \mathbb{R}$

s.t.

- $u_{s+t} = u_s \sigma_s^{\psi}(u_t)$, $s, t \in \mathbb{R}$
  (σ^{ψ}– cocycle identity)
- $\sigma_t^{\varphi} = \mathrm{Ad}\, u_t \circ \sigma_t^{\psi}$
- Relative KMS condition.

$$[D\varphi : D\psi]_t := u_t$$

☆

- $[D\varphi : D\psi]_t = 1 \ \forall t \iff \varphi = \psi$
- $[D\varphi : D\psi]_t [D\psi : D\chi]_t = [D\varphi : D\chi]_t$
- If u_t is a σ^{ψ}– cocycle,
  then $\exists ! \varphi \in W(M)$ s.t. $u_t = [D\varphi : D\psi]_t$.

§1.3 Standard form.

(M, H, J, P) is std form

vN alg $\curvearrowright H$ Hilb   conj-~~linear~~ unitary involution. $JMJ = M'$

P : positive cone (norm-closed)

- "self-dual"

i.e. If $\eta \in H$ satisfies

$$\langle \eta, \xi \rangle \geq 0 \quad \forall \xi \in P$$

then $\eta \in P$.

- $J\xi = \xi \quad \forall \xi \in P$
- $xJxJ\xi \in P$

Ex. 1.4

$(M, H_\varphi, J_\varphi, P_\varphi)$ is std.

$$P_\varphi := \overline{\{ x J_\varphi \Lambda_\varphi(x) \mid x \in n_\varphi \cap n_\varphi^* \}}^{\text{closure}}$$

Thm. 1.5

A std form of M is unique!

i.e.

(π_1, H_1, J_1, P_1)
(π_2, H_2, J_2, P_2) : std's

then $\exists!$ $U : H_1 \longrightarrow H_2$ unitary

s.t.

- $U\pi_1(x) = \pi_2(x)U, \quad \forall x \in M$
- $UJ_1 = J_2 U$
- $UP_1 = P_2$

Cor. 1.6 (M, H, J, P) std.

$\forall \alpha \in \mathrm{Aut}(M)$.

$\exists! \, U_\alpha \in \mathcal{U}(H)$

s.t.

- $\alpha(x) = U_\alpha x U_\alpha^*$, $\forall x \in M$
- $U_\alpha J = J U_\alpha$
- $U_\alpha P = P$.

Thm. 1.7

$\forall \omega \in M_*^+$

$\exists! \, \xi \in P$

s.t. $\omega = \langle \cdot \, \xi, \xi \rangle$

Denote it by ξ_ω

☆ Araki – Powers – Størmer ineq.

$$\| \xi_{\omega_1} - \xi_{\omega_2} \|^2 \leq \| \omega_1 - \omega_2 \| \leq \| \xi_{\omega_1} - \xi_{\omega_2} \| \| \xi_{\omega_1} + \xi_{\omega_2} \|$$

☆ u-topology

$\alpha_\lambda \xrightarrow{u} \alpha$ in $\mathrm{Aut}(M)$

$\overset{\text{defn}}{\iff} \alpha_\lambda(\varphi) \to \alpha(\varphi) \quad \forall \varphi \in M_*$

- $\alpha \in \mathrm{Aut}(M)$ $\quad \omega = \langle \cdot \, \xi_\omega, \xi_\omega \rangle$, $\xi_\omega \in P$

$$\alpha(\omega)(x) = \omega(\alpha^{-1}(x))$$
$$= \langle \alpha^{-1}(x) \xi_\omega, \xi_\omega \rangle$$
$$= \langle U_\alpha^* x U_\alpha \xi_\omega, \xi_\omega \rangle$$
$$= \langle x \, U_\alpha \xi_\omega, U_\alpha \xi_\omega \rangle$$

$U_\alpha \xi_\omega \in P$

$\rightsquigarrow \xi_{\alpha(\omega)} = U_\alpha \xi_\omega$

$\rightsquigarrow \alpha_\lambda \xrightarrow{u} \alpha \overset{\text{APS}}{\iff} U_{\alpha_\lambda} \xrightarrow{\text{strong}} U_\alpha$

<u>Notation</u>.

- $x \xi y := xJy^*J\xi \rightsquigarrow xPx^* \subset P$. $\left(\overset{\shortparallel}{xJxJP}\right)$

  $x, y \in M$

  $\xi \in H$.

  (bimodule property)

- $\alpha(\xi) := U_\alpha \xi$

  $\alpha \in \mathrm{Aut}(M)$

  $\xi \in H$

$\triangleright$ If $\alpha = \mathrm{Ad}\,u$ $(u \in M^u)$

then $U_\alpha = uJuJ$

$\rightsquigarrow \alpha(\xi) = uJuJ\xi$

$= u\xi u^*$.

<u>Lem</u>. 1.8

. $e \in M^P$

$\rightsquigarrow (eMe, eHe, eJe, ePe)$ std.

<u>proof</u> Easy

<u>Thm</u> 1.9 (Connes)

$\xi \in P$. TFAE.

(1) ξ cyclic

(2) ξ sep.

(3) If $\eta \in P$ & $\eta \perp \xi$

then $\eta = 0$

proof

(1) $\Longleftrightarrow$ (2) $\quad J\xi = \xi$.

$$M'\xi = JMJ\xi$$
$$= JM\xi$$

(1),(2) $\Longrightarrow$ (3) $\quad \xi$: cyc & sep $\in \mathcal{P}$

$$\mathcal{P}_\xi := \overline{\Delta_\varphi^{\frac{1}{4}} M_+ \xi}$$

where $\varphi = \langle \cdot\, \xi, \xi \rangle \in M_*^+$.

$\rightsquigarrow \mathcal{P} = \mathcal{P}_\xi$.

Let $x \in M_+$. Then

$$\underbrace{\Delta_\varphi^{\frac{1}{4}} x \xi}_{\in\, \mathcal{P}_\xi} + \underbrace{\Delta_\varphi^{\frac{1}{4}} (\underbrace{\|x\| - x}_{\geq 0}) \xi}_{\in\, \mathcal{P}_\xi} = \|x\| \xi$$

$$\underbrace{\langle \Delta_\varphi^{\frac{1}{4}} x \xi, \eta \rangle}_{\geq 0} + \underbrace{\langle \Delta_\varphi^{\frac{1}{4}} (\|x\| - x) \xi, \eta \rangle}_{\geq 0} = \|x\| \cdot \underbrace{\langle \xi, \eta \rangle}_{= 0}.$$

$\downarrow$ limit

η

(3) $\Longrightarrow$ (1), (2)

Let $e \in M^P$ & $e\xi = 0$.

$$\langle \underset{\in\, \mathcal{P}}{\underbrace{e\overset{\forall}{\eta}e}}, \xi \rangle = \langle \eta, e\xi e \rangle = 0$$

By (3) $\quad \underbrace{e\eta e}_{= eJeJ\eta} = 0 \quad \forall \eta \in \mathcal{P}$

$\to eJeJ = 0 \quad$ ($\mathcal{P}$ generates H)

$\to MeM\, JeJ = 0$

$\to Z_M(e)\, JeJ = 0$

$$Z_M(e)\, JeJ = J Z_M(e)\, eJ = JeJ$$

$\to e = 0$

$\blacksquare$

<u>Cor.</u> 1.11

$\xi \in \mathcal{P}$ cyc & sep for M

For $\forall e \in M^P$

$\Longrightarrow$ $e\xi e \in e\mathcal{P}e$ cyc & sep for eMe

<u>Proof</u>.

Let $\eta \in e\mathcal{P}e$ & $\langle \eta, e\xi e \rangle = 0$

$$\begin{array}{c} \langle \eta, e\xi e \rangle \\ \| \\ \langle \underset{\substack{\cap \\ P}}{e\eta e}, \underset{\substack{\cap \\ 1}}{\xi} \rangle \end{array} \qquad (\eta = e\eta e)$$

By prev thm, 1.9 $\eta = e\eta e = 0$

Again by prev. thm 1.9 $e\xi e =$ cyc & sep.

Prop. 1.10

For $(x_n) \in \ell^\infty(\mathbb{N}, M)$, TFAE.
& $\varphi \in M_*$ state.

(1) $\lim_{n\to\infty} \|x_n \varphi - \varphi x_n\| = 0$

(2) $\lim_{n\to\infty} \|x_n \xi_\varphi - \xi_\varphi x_n\| = 0$

proof

$A := \{ (x_n) \mid (x_n) \text{ satisfies } (1) \}$

$B := \{ \quad \mid \quad (2) \}$

$A, B \subset \ell^\infty(\mathbb{N}, M)$ unital C^*-subalg

Suffices to show $A^u = B^u$.

Then $\forall (u_n) \in \ell^\infty(\mathbb{N}, M)^u$, we have

by Araki - Powers - Størmer ineq.

$$\|u_n \xi_\varphi u_n^* - \xi_\varphi\|^2 \leq \|u_n \varphi u_n^* - \varphi\| \leq \|u_n \xi_\varphi u_n^* - \xi_\varphi\| \times 2$$

[illegible]

§1.4 Actions & Crossed Products

G : loc. cpct. group

$\alpha : G \longrightarrow \mathrm{Aut}(M)$ action

$\overset{\text{defn}}{\iff}$ • $\alpha_{st} = \alpha_s \circ \alpha_t \quad \forall s,t \in G$

• continuous wrt the u-top.

Let L^2G wrt the Haar measure.

$(\lambda(s) f)(t) := f(s^{-1}t)$ left

$(\rho(s) f)(t) := f(ts)$ right

regular repn's

$LG := \lambda(G)''$

$RG := \rho(G)''$

commutant

Crossed Products

$G \overset{\alpha}{\curvearrowright} M$ action

$\rightsquigarrow M \rtimes_\alpha G \curvearrowright H \otimes L^2G$

$\overset{\text{vN.}}{\|}$

$\pi_\alpha(M) \vee \lambda^\alpha(G)''$
$\qquad (\mathbb{C} \otimes LG)$

$(\pi_\alpha(x)\,\xi)(s) = \alpha_s^{-1}(x)\,\xi(s)$

$\xi \in H \otimes L^2G = L^2(G, H)$

$(\underset{\underset{1 \otimes \lambda(t)}{\|}}{\lambda^\alpha(t)}\,\xi)(s) = \xi(t^{-1}s)$

☆ $M \rtimes_\alpha G \supset \pi_\alpha(M)$ is cocycle conj-invariant

i.e.

If v : α-cocycle

$$\underset{\substack{\cap \\ M^u}}{v_{st}} = v_s \alpha_s(v_t)$$

then $\exists$.

$$\begin{array}{ccc} M \rtimes_\alpha G & \cong & M \rtimes_{\alpha^u} G \\ \cup & & \cup \\ \pi_\alpha(M) & & \pi_{\alpha^u}(M) \\ x & \longmapsto & x \end{array}$$

$$\alpha^u_s \cancel{(x)} = \mathrm{Ad}\, v_s \circ \alpha_s$$

$$\pi_\alpha(v_s)\lambda^\alpha(s) \longleftrightarrow \lambda^{\alpha^u}(s)$$

☆ Duality

G : abelian loc. cpct.

$G \overset{\alpha}{\curvearrowright} M \rightsquigarrow M \rtimes_\alpha G \overset{\hat{\alpha}}{\curvearrowleft} \hat{G}$

$$\begin{cases} \cdot\ \hat{\alpha}_p(\pi_\alpha(x)) = \pi_\alpha(x) & \forall p \in \hat{G} \\ \cdot\ \hat{\alpha}_p(\lambda^\alpha(s)) = \overline{\langle s, p\rangle}\, \lambda^\alpha(s). \end{cases}$$

$\rightsquigarrow (M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$

Thm. 1.12 (Takesaki)

$$(M \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G} \cong M \bar{\otimes} B(L^2 G)$$

$$\pi_{\hat{\alpha}}(\pi_\alpha(x)) \longleftrightarrow \pi_\alpha(x)$$

$$\pi_{\hat{\alpha}}(\lambda^\alpha(s)) \longleftrightarrow \lambda^\alpha(s) = 1 \otimes \lambda(s)$$

$$\lambda^{\hat{\alpha}}(p) \longleftrightarrow 1 \otimes \underset{\in L^\infty(G)}{\mu(p)}$$

$$\mu(p)(s) := \overline{\langle s, p\rangle}$$

(M, H, J, P) std.

$\rightsquigarrow (M \rtimes_\alpha G, H \otimes L^2G, \tilde{J}, \hat{P})$ std. (loc. cpct. G)

$$(\tilde{J}\xi)(s) := \Delta_G \, U^\alpha_{s^{-1}} J \xi(s^{-1})$$

$$\rightsquigarrow \begin{cases} \cdot\ \tilde{J}\, \pi_\alpha(x)\, \tilde{J} = JxJ \otimes 1 \\ \cdot\ \tilde{J}\, \lambda^\alpha(s)\, \tilde{J} = U_\alpha(s) \otimes \rho(s) \end{cases}$$

$$\rightsquigarrow (M \rtimes_\alpha G)' = \tilde{J}(M \rtimes_\alpha G)\tilde{J}$$

$$= (M' \otimes \mathbb{C}) \vee \{U_\alpha(s) \otimes \rho(s)\}''_{s \in G}$$

commutant

$$M \rtimes_\alpha G = (M \otimes B(L^2G)) \cap \{U_\alpha(s) \otimes \rho(s)\}'_s$$

$$= (M \otimes B(L^2G))^{\alpha \otimes \mathrm{Ad}\rho}$$

"fixed pt characterization"

§1.5 Type III vN algs

M is of type III

$\overset{\text{defn}}{\iff}$ $\forall e \neq 0 \in M^P$ is not finite

$\overset{\sigma\text{-fin}}{\iff}$ $\forall e \in M^P \quad e \sim Z_M(e)$

$\overset{\sigma\text{-fin \& factor}}{\iff}$ $\forall \underset{0}{e \neq} \in M^P \quad e \sim 1.$

$\varphi \in W(M).$

$$(M \rtimes_{\sigma^\varphi} \mathbb{R}) \rtimes_{\widehat{\sigma^\varphi}} \mathbb{R} \cong M \otimes B(L^2\mathbb{R}) \cong M$$

when prop inf

Thm. 1.13 (Takesaki)

M type III vN alg.

(1) $\exists !$ $\{N, \theta, \mathbb{R}\}$

up to conjugacy, type II_∞, trace-scaling flow

$\tau \circ \theta_t = e^{-t}\tau \quad \forall t \in \mathbb{R}$

tracial wt.

s.t.

$M \cong N \rtimes_\theta \mathbb{R}$

(2) $Z(M) \cong Z(N)^\theta.$

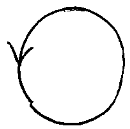
Let M type III factor

$$\mathcal{Z}(N)^{\theta} = \mathcal{Z}(M) = \mathbb{C}$$

i.e. $\mathbb{R} \overset{\theta}{\curvearrowright} \mathcal{Z}(N)$ ergodic.

$$\ker\theta := \{ t \in \mathbb{R} \mid \theta_t = \mathrm{id} \text{ on } \mathcal{Z}(N) \}$$

$$= \begin{cases} \cdot\ T\mathbb{Z} & \text{Type III}_{\lambda} \ (\lambda = e^{-T}) \\ \cdot\ 0 & \text{Type III}_{0} \\ \cdot\ \mathbb{R} & \text{Type III}_{1} \end{cases}$$



Type III_1 $\Updownarrow$ N type II_{∞}-factor.

1.6

§ Arveson spectrum.

$\mathbb{R} \overset{\alpha}{\curvearrowright} M$ action.

$x \in M$

$I_x := \{ f \in L^1\mathbb{R} \mid \alpha_f(x) = 0 \}$

$\triangleleft L^1\mathbb{R}$ closed ideal.

$$Sp_\alpha(x) := \{ p \in \mathbb{R} \mid \hat{f}(p) = 0 \quad \forall f \in I_x \}$$

where

$$\hat{f}(p) = \int_{\mathbb{R}} f(t)\, e^{ipt}\, dt$$

☆ If $\alpha_t(x) = e^{ipt} x$. $\forall t$

$\rightarrow$ $Sp_\alpha(x) = \{p\}$

prop. 1.17

$\mathbb{R} \overset{\alpha}{\curvearrowright} M$.

$x \in M$, $\delta > 0$ given.

T.F.A.E.

(1) $Sp_\alpha(x) \subset [-\delta, \delta]$

(2) x is analytic of exponential type

$$\|\alpha_z(x)\| \leq K\, e^{\delta |\mathrm{Im}\, z|} \quad \forall z \in \mathbb{C}$$

for some $K > 0$

proof

(2) $\Rightarrow$ (1). $\varphi \in M_*$ fixed.

$$|\varphi(\alpha_z(x))| \leq K \|\varphi\|\, e^{\delta |\mathrm{Im}\, z|}, \quad \forall z \in \mathbb{C}$$

By Paley–Wiener's thm.

$\exists H$ tempered distribution

$\operatorname{supp} H \subset [-\delta, \delta]$ cpct support

s.t. $\varphi(\alpha_t(x)) = \hat{H}(t)$. $\forall t$

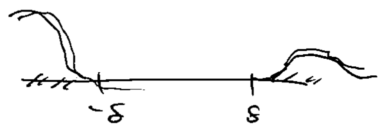
(1) $\Longrightarrow$ (2) 逆に証明できる

Then, for $f \in \mathcal{S}(\mathbb{R})$, we have

$$\varphi(\alpha_f(x)) = \int_{\mathbb{R}} dt\, f(t)\, \varphi(\alpha_t(x))$$

$$= \langle f, \hat{H} \rangle$$

$$= \langle \hat{f}, H \rangle .$$

So, if $\operatorname{supp} \hat{f} \subset$ 

then $\varphi(\alpha_f(x)) = 0$. $\forall \varphi \in M_*$

$\rightsquigarrow$ $\alpha_f(x) = 0$

$\rightsquigarrow$ $\operatorname{Sp}_\alpha(x) \subset [-\delta, \delta]$.

Section 2

§2.1 Connes – Takesaki: relative comm. theorem.

Thm ~~1.14~~ 2.1

$$\boxed{\begin{array}{l} M : \text{Type III} \\ \qquad M = N \rtimes_\theta \mathbb{R} \quad \text{cont. decp.} \\ \Rightarrow \pi_\theta(N)' \cap M = Z(N) \end{array}}$$

Let $\varphi \in W(M)$ &

$$\widetilde{M} := M \rtimes_{\sigma^\varphi} \mathbb{R}$$

then

$$\begin{array}{ccc} \widetilde{M} \rtimes_\theta \mathbb{R} & \overset{\text{duality}}{\cong} & M \otimes B(L^2\mathbb{R}). \\ \cup \ {\scriptstyle \| \atop \widehat{\sigma^\varphi}} & & \cup \\ \pi_\theta(\widetilde{M}) & \longleftrightarrow & \widetilde{M} \end{array}$$

Thus want to show

$$\widetilde{M}' \cap (M \otimes B(L^2\mathbb{R})) = Z(\widetilde{M}) \qquad - (*)$$

☆ Applying $\widetilde{J} = J_{\widetilde{M}}$ we have

$$\widetilde{M} \cap \widetilde{J}(M \otimes B(L^2\mathbb{R}))\widetilde{J} \overset{?}{=} Z(\widetilde{M})$$
$$\|$$
$$\widetilde{M} \cap \pi_{\sigma^\varphi}(M)'$$

Recall

$$\widetilde{M} = M \rtimes_{\sigma^\varphi} \mathbb{R}$$
$$= \pi_{\sigma^\varphi}(M) \vee \mathbb{C} \otimes L\mathbb{R}.$$

Thus

$$\text{LHS of } (*) = \pi_{\sigma^\varphi}(M)' \cap (M \otimes L\mathbb{R})$$

$$(*) \Leftrightarrow \pi_{\sigma^\varphi}(M)' \cap (M \otimes L\mathbb{R}) \overset{?}{\subset} \widetilde{M}$$

Btw.

$$(M\otimes L\mathbb{R})\cap\widetilde{M} = (M\otimes L\mathbb{R}) \quad \alpha^{\varphi}\otimes \mathrm{Ad}\rho$$

commute.

$$= M_{\varphi}\otimes L\mathbb{R}$$

Thus we want to show.

$$\pi_{\sigma^\varphi}(M)'\cap(M\otimes L\mathbb{R})\subset M_\varphi\otimes L\mathbb{R} \quad -(**)$$

Let $F: L^2\mathbb{R}\to L^2\mathbb{R}$ Fourier trans.

$$\begin{array}{ccc} \cup & & \in \\ f & \longmapsto & Ff(p) = \int_{\mathbb{R}} f(x)\, e^{-ipx}\,dx \end{array}$$

$dx := \frac{1}{\sqrt{2\pi}}$ Lebesgue

$$\widetilde{F} : H\otimes L^2\mathbb{R}\to H\otimes L^2\mathbb{R}$$

$$\begin{array}{c} \| \\ 1_H\otimes F \end{array}$$

$(**)$

$$\Longleftrightarrow \widetilde{F}\pi_{\sigma^\varphi}(M)\widetilde{F}^{*\prime}\cap(M\otimes L^\infty\mathbb{R})\subset M_\varphi\otimes L^\infty\mathbb{R}$$

$$\| \quad \text{(disintegration)}.$$

$$L^\infty(\mathbb{R}, M) \qquad (***)$$

Note

$$\widetilde{F}\pi_{\sigma^\varphi}(M)\widetilde{F}^* \subset \widetilde{F}(M\otimes L^\infty\mathbb{R})\widetilde{F}^*$$

$$\subset M\otimes L\mathbb{R}.$$

Thus,

$\theta_t := \mathrm{Ad}(1\otimes\lambda(t))$ fixes $\widetilde{F}\pi_{\sigma^\varphi}(M)\widetilde{F}^*$.

$$\rightsquigarrow \mathbb{R}\overset{\theta}{\curvearrowright}\widetilde{F}\pi_{\sigma^\varphi}(M)\widetilde{F}^{*}\cap(M\otimes L^\infty(\mathbb{R}))$$

Let

$$C(\mathbb{R}, M) := \left\{ x \;\middle|\; \begin{array}{l} x:\mathbb{R}\to M \\ s^*\text{- cont} \\ \text{norm bdd.} \end{array} \right\}$$

Lem. ~~1.15~~ 2.2

If
$$\widetilde{F}\pi_{\varphi}(M)\widetilde{F}^{*\prime} \cap C(\mathbb{R}, M) \subset C(\mathbb{R}, M_\varphi)$$
then $(***)$ holds. (Thus Thm 2.1 holds)

<u>proof</u>

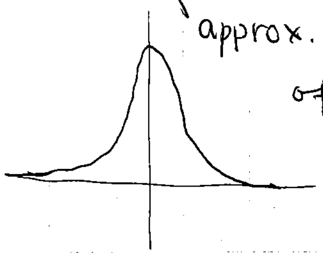
Let $f \in C_c(\mathbb{R})$ & $x \in L^\infty(\mathbb{R}, M)$

$$\theta_f(x) := \int_{\mathbb{R}} f(t)\,\theta_t(x)\,dt \in C(\mathbb{R}, M).$$

$$\left(\theta_s(\theta_f(x)) = \theta_{\lambda_s f}(x) \quad \lambda_s f \xrightarrow{L^1} f \right)$$

$$\theta_{f_i}(x) \xrightarrow{s^*} x$$

f_i: approx. unit of $L^1(\mathbb{R})$.



Now let $x \in$ LHS of $(***)$.

Then. $\theta_{f_i}(x) \in \widetilde{F}\pi_{\varphi}(M)\widetilde{F}^{*\prime} \cap C(\mathbb{R}, M)$, $\theta_{f_i}(x) \to x$

$\cap$ by our assumption

$C(\mathbb{R}, M_\varphi)$

$\cap$

$M_\varphi \otimes L^\infty(\mathbb{R})$. ∎

From now, we may & do assume that φ is a faithful normal state.

Lem. ~~1.16~~ 2.3

$a \in C(\mathbb{R}, M)$

$x \in M$

$f, g \in C_c^\infty(\mathbb{R})$

Then

(1) $\langle \tilde{F} \pi_\varphi(x) \tilde{F}^*(\xi_\varphi \otimes f), a^*(\xi_\varphi \otimes g) \rangle$

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dp\, dq\; e^{-it(p-q)} f(p) \overline{g(q)} \cdot \varphi(a(q) \sigma_t^\varphi(x))$$

(2) If $a \in \tilde{F} \pi_\varphi(x) \tilde{F}^{*\prime}$

then

$$\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} e^{-it(p-q)} f(p) f(q)\, \varphi(a(q) \sigma_t^\varphi(x))$$

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} e^{p-q} e^{-it(p-q)} f(p) g(q)\, \varphi(a(p) \sigma_t^\varphi(x))$$

Proof

(1)

$$\underset{L^2(\mathbb{R}, H)}{\left[\pi_\varphi(x) \tilde{F}^*(\xi_\varphi \otimes f)\right](t)}$$

$$= \left(\sigma_{-t}^\varphi(x)\, \xi_\varphi \otimes F^* f(t)\right)$$

$\in \mathbb{C}$, $F^* f \in \mathcal{S}(\mathbb{R})$

$$[\tilde{F} \pi_\varphi(x) \tilde{F}^*(\xi_\varphi \otimes f)](q)$$

$$= \int_{\mathbb{R}} dt\; e^{-itq} [\pi_\varphi(x) \tilde{F}^*(\xi_\varphi \otimes f)](t)$$

$$= \int_{\mathbb{R}} dt\; e^{-itq} (F^* f)(t)\; \sigma_{-t}^\varphi(x) \xi_\varphi$$

Hence.

$$\langle \widetilde{F}\pi_\sigma\varphi(x)\widetilde{F}^*(\xi_\varphi\circ f),\ a^*(\xi_\varphi\circ g)\rangle$$

$$= \int_{\mathbb{R}} dq \langle [\widetilde{F}\pi_\sigma\varphi(x)\widetilde{F}^*(\xi_\varphi\circ f)](q),\ a(q)^*\xi_\varphi\rangle \cdot \overline{g(q)}$$

$$= \int_{\mathbb{R}} dq\ \underset{C_c^\infty(\mathbb{R})}{\overline{g(q)}} \int_{\mathbb{R}} dt\ e^{-itq} \underset{\mathcal{S}(\mathbb{R})}{(F^*f)(t)} \underbrace{\langle \sigma_t^\varphi(x)\xi_\varphi, a(q)^*\xi_\varphi\rangle}_{\text{bdd cont.}}$$

Fubini

$$= \int_{\mathbb{R}} dt\ \underline{(F^*f)(t)} \int dq\ \overline{g(q)}\, e^{-itq}\ \varphi(a(q)\sigma_t^\varphi(x))$$

$$\| $$

$$\int dp\ \underset{C_c^\infty(\mathbb{R})}{f(p)}\, e^{itp}$$

Fubini

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dp\,dq\ e^{it(p-q)} f(p)\,\overline{g(q)}\ \varphi(a(q)\sigma_t^\varphi(x))$$

$$t \longrightarrow -t.$$

(1)

(2)

$$\langle \widetilde{F}\pi_\sigma\varphi(x)\widetilde{F}^*(\xi_\varphi\circ f),\ a^*(\xi_\varphi\circ \overline{g})\rangle =: A$$

$$= \overline{\langle \widetilde{F}\pi_\sigma\varphi(x^*)\widetilde{F}^*(\xi_\varphi\circ \overline{g}),\ a(\xi_\varphi\circ f)\rangle} =: B$$

By (1)

$$A = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} e^{-it(p-q)} f(p)\, g(q) \cdot \varphi(a(q)\sigma_t^\varphi(x)).$$

$$B = \overline{\int_{\mathbb{R}} dt \int_{\mathbb{R}^2} e^{-it(p-q)}\, \overline{g(p)}\ \overline{f(q)} \cdot \varphi(a(q)^*\sigma_t^\varphi(x^*))}$$

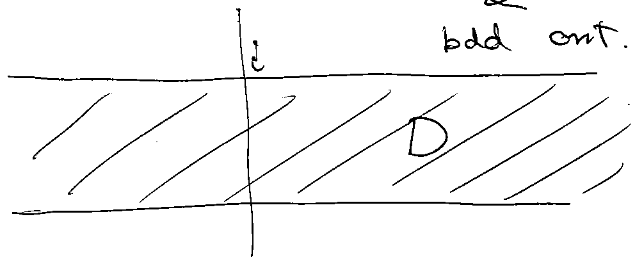
$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} e^{it(p-q)} g(p)\, f(q) \cdot \varphi(\sigma_t^\varphi(x)\, a(q))$$

$$p \longleftrightarrow q$$

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dp\,dq\ e^{-it(p-q)} f(p)\, g(q)\ \varphi(\sigma_t^\varphi(x)\, a(p))$$

By KMS cond. (Thm 1,2)

For each $p \in \mathbb{R}$, $\exists\, G_p(z)$ hol. on D & bdd cont. on $\overline{D}$



s.t.

$$G_p(t) = \varphi(\sigma_t^\varphi(x)\, a(p))$$

$$G_p(t+i) = \varphi(a(p)\, \sigma_t^\varphi(x)) \qquad \forall t \in \mathbb{R}.$$

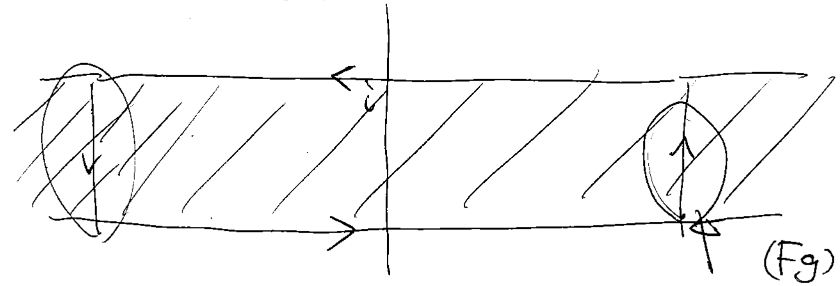
Then

$$B = \int_{\mathbb{R}} dt \int_{\mathbb{R}^2} dp\,dq\ \underbrace{e^{-it(p-q)}}\ \underset{f(p)g(q)}{G_p(t)}$$

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}} dp\ e^{-itp}\ \underset{\cap}{f(p)}\ \underset{\underset{\mathcal{S}(\mathbb{R})}{\uparrow}}{(Fg)(-t)}\ \underset{\underset{C_b(\mathbb{R}^2)}{\uparrow}}{G_p(t)}$$

$$= \int_{\mathbb{R}} dp \int_{\mathbb{R}} dt\ f(p)\ \underbrace{e^{-itp}(Fg)(-t)\, G_p(t)}_{\substack{\| \\ H_p(t)}}$$

$$H_p(z) = \underset{\text{entire}}{e^{-izp}\,(Fg)(-z)}\ G_p(z)$$



$(Fg)(-z)$ decay fast. for each p.

$$= \int_{\mathbb{R}} dp \int_{\mathbb{R}} dt\ f(p)\ e^{-i(t+i)p}\ \underset{\underset{\mathcal{S}(\mathbb{R})}{\cup}}{(Fg)(-t-i)}\ G_p(t+i)$$

Fubini

$$= \int_{\mathbb{R}} dt \int_{\mathbb{R}} dp\ f(p)\ e^{p}\ e^{-itp}\ \overset{\int g(q)\, e^{itq}\, e^{-q}}{\overset{\|}{(Fg)(-t-i)}}$$

$$\cdot\ \varphi(a(p)\, \sigma_t^\varphi(x))$$

$\blacksquare$

<u>Lem.</u> I.II 2.4

Suppose $H \in C_b(\mathbb{R}^2)$ satisfies.

$$\int dt \int dp\,dq\; e^{-it(p-q)} f(p)\, g(q)\, H(t,q)$$

$$= \int dt \int dp\,dq\; e^{p-q}\, e^{-it(p-q)} f(p)\, g(q)\, H(t,p)$$

for $\forall\, f, g \in C_c^\infty(\mathbb{R})$.

Then

$$H(t,p) = H(0,p) \quad \forall (t,p) \in \mathbb{R}^2$$

<u>Proof of Thm</u> I.II 2.1

Let $a \in \tilde{F}\pi_\varphi(M)\tilde{F}^{*\prime} \cap C(\mathbb{R}, M)$.

Putting $H(t,p) := \varphi(a(p)\,\sigma_t^\varphi(x))$.

We have

$$H(t,p) = H(0,p) \quad \forall (t,p) \in \mathbb{R}^2$$

by Lem I.II 2.4.

This implies

$$\varphi(a(p)\,\sigma_t^\varphi(x)) = \varphi(a(p)\,x) \quad \forall x \in M,\ p \in \mathbb{R}.$$

$\rightarrow$ $G_p(z) = \text{const}$

$= G_p(t+i)$

$= \varphi(\sigma_t^\varphi(x)\, a(p)) \quad \forall x \in M$

$\rightarrow$ $a(p) \in M_\varphi \quad \forall p \in \mathbb{R}.$

$\blacksquare$

$$\text{LHS} = \varphi(\sigma_t^\varphi(a(p))\, x) \quad \forall x$$

$\rightsquigarrow$ $\sigma_t^\varphi(a(p)) = a(p)$

$\rightsquigarrow$ $a(p) \in M_\varphi \quad \forall p.$

$\blacksquare$

<u>proof of Lem ~~1.17~~ 2.4</u>

For $f \in C_c^\infty(\mathbb{R}^2)$, we set

$$\langle f, \widetilde{H_1} \rangle := \int dt \int dp\,dq\; e^{-it(p-q)} \cancel{f(p)\,g(q)}\, H(t,q)$$
$$f(p,q)$$

$$\langle f, \widetilde{H_2} \rangle := \int dt \int dp\,dq\; e^{p-q}\, e^{-it(p-q)} \cancel{f(p)\,g(q)}$$
$$f(p,q)$$
$$H(t,p)$$

$\widetilde{H_1}$, $\widetilde{H_2}$ are distributions.

Since $C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathbb{R}) \subset C_c^\infty(\mathbb{R}^2)$ dense

(Fourier series & $e^{i(\cancel{p}x+y)} = e^{ix}e^{iy}$)

We have $\widetilde{H_1} = \widetilde{H_2}$.

Let $f \in C_c^\infty(\mathbb{R}^2)$

Then $\langle f, \widetilde{H_1} \rangle = \langle f, \widetilde{H_2} \rangle$

$$\overset{p \to p+q}{\Leftrightarrow} \int dt \int dp\,dq\; e^{-itp} f(p+q,q)\, H(t,q)$$

$$= \int dt \int dp\,dq\; e^{p}\, e^{-itp} f(p+q,q)\, H(\cancel{p+q}\,t)$$

t. $p \in q$

$\forall f \in C_c^\infty(\mathbb{R}^2)$

$(p,q) \to (p+q, q)$ bijective linear

replace

$$\Leftrightarrow \int dt \int dp\,dq\; e^{-itp} f(p,q)\, H(t,q)$$

$$= \int dt \int dp\,dq\; e^{p}\, e^{-itp} f(p,q)\, H(\overset{t,}{p+q}\,\cancel{q})$$

$$\| \quad q \to q-p$$

$$\int dt \int dp\,dq\; e^{p}\, e^{-itp} f(p, q-p)\, H(t, \cancel{p+q}\, q)$$

i.e.

$$\int dt \int dp\,dq\; e^{-itp} \left(f(p,q) - e^{p} f(p,q-p) \right) H(t,q)$$
$$= 0$$
$$\text{— } (*)$$

Then put for $f \in \mathcal{S}(\mathbb{R}^2)$

$$\langle f, \widehat{H} \rangle := \int dt \int dp\,dq\; e^{-itp} f(p,q) H(t,q)$$

$$= \int dt\, (\overset{\text{第1変数}}{F_1} f)(t,q) H(t,q)$$

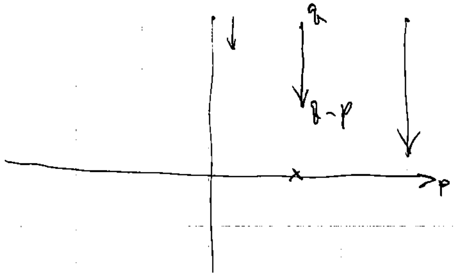
$$= \langle \underset{\in \mathcal{S}(\mathbb{R}^2)}{F_1 f}, \underset{\in C_b(\mathbb{R}^2)}{H} \rangle \quad \text{tempered distr.}$$

$$\underset{\text{by definition}}{=} \langle f, F_1 H \rangle$$

i.e. $\widehat{H} = F_1 H \in \mathcal{S}'(\mathbb{R}^2)$.

Let $T \circlearrowright \cancel{\mathcal{S}(\mathbb{R}^2)}\ C_c^\infty(\mathbb{R}^2)$

$$(Tg)(p,q) := e^p g(p, q-p)$$



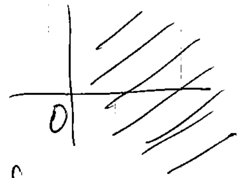
By $(*)$, we have

$$\langle (1-T)f, \widehat{H} \rangle = 0 \quad \forall f \in C_c^\infty(\mathbb{R}^2)$$

Claim. $\operatorname{Supp} \widetilde{H} \subset \{0\} \times \mathbb{R}$.

proof

$f \in C_c^\infty(\mathbb{R}^2)$ & $\operatorname{supp} f \subset$



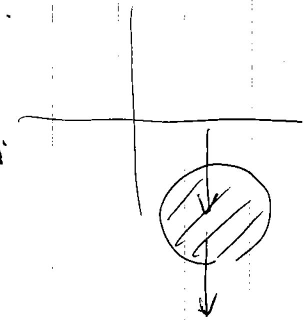
$\{p > 0\}$.

$$g_n \cancel{(p,q)} := -\sum_{k=1}^{n} (T^{-k} f) \cancel{(p,q)} \in C_c^\infty(\mathbb{R}^2).$$

$$(1-T) g_n = -T^{-n} f + f$$

$$\xrightarrow[n \to \infty]{} f \quad \text{in } \mathcal{S}(\mathbb{R}^2).$$

$$(T^{-n} f)(p,q) = e^{-np} \underbrace{f(p, q+np)}.$$



$$p_N(f) := \sup_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^2} (1+|x|^2)^N |(\partial^\alpha f)(x)|$$

for each $N \in \mathbb{N}$.

$$p_N(T^{-n} f) \leq e^{-n\varepsilon_0} \times R^{2N} \times \underset{\text{depends on } R \,\&\, N}{C} \times n^N \quad \forall n$$

$\operatorname{supp} f \subset [\varepsilon_0, R] \times [-R, R]$

$\to 0$.

Thus

$$\langle f, \widetilde{H} \rangle = \lim_{n \to \infty} \langle (1-T) g_n, \widetilde{H} \rangle$$

$$= 0$$

Similarly, we can show

$\langle f, \widetilde{H} \rangle = 0$ if $\operatorname{supp} f \subset (-\infty, 0) \times \mathbb{R}$

Hence $\operatorname{supp} \widetilde{H} \subset \{0\} \times \mathbb{R}$

Claim. $\boxed{}$

Hence, if $f, g \in C_c^\infty(\mathbb{R})$ & $0 \notin \operatorname{supp} f$,

then,

$$0 = \langle f \otimes g, \widetilde{H} \rangle$$

$$= \langle f \otimes g, F_1 H \rangle_{\mathcal{S}(\mathbb{R}^2)}$$

$$= \langle F_1 f \otimes g, H \rangle_{\mathcal{S}(\mathbb{R}^2)}$$

$$= \int dt\, dq\, F_1 f(t)\, g(q)\, H(t,q)$$

$$\overset{\text{Fubini}}{=} \int dq\, g(q) \underbrace{\int dt\, \overset{\mathcal{S}(\mathbb{R})}{\overset{\Downarrow}{F_1 f}}(t)\, H(t,q)}_{\in\, C_b(\mathbb{R}_q)}$$

$$\rightsquigarrow \int dt\, F f(t)\, H(t,q) = 0 \quad \forall q \in \mathbb{R}$$

$\parallel$ q fixed

$$\langle f, F_1 H(\cdot, q) \rangle_{\mathcal{S}(\mathbb{R})}$$

$\mathcal{S}'(\mathbb{R})$

$\rightsquigarrow \quad \operatorname{supp} F_1 H(\;, q) \subset \{0\}$

Thus $F_1 H(p, q) = \displaystyle\sum_{k=1}^{m_q} \alpha_q^k \frac{d^k}{dt^k} \delta_0(p)$, $\quad \alpha_q^k \in \mathbb{C}$

$\rightsquigarrow \quad H(t, q) = \displaystyle\sum_{k=1}^{m_q} \tilde{\alpha}_q^k \, t^k$

bdd cont. $\qquad$ polynomial

$\overset{\curvearrowright}{=} \tilde{\alpha}_q^0 \, 1$

indep. of t.

Lem ~~1.17~~ 2.4 $\blacksquare$

§2.2

Integrable action & dominant wt

$G \overset{\alpha}{\curvearrowright} M$ action

<u>integrable</u>

$\overset{\text{defn}}{\Longleftrightarrow}$ $\mathfrak{m}^+_{T_\alpha} \subset M_+$ s-dense,

where

$$\mathfrak{m}^+_{T_\alpha} := \left\{ x \in M_+ \;\middle|\; \underset{\text{bdd operator}}{\int_G \alpha_t(x)\,dt} \right\}$$

$$T_\alpha(x) := \int_G \alpha_t(x)\,dt$$

☆ $T_\alpha : M_+ \to \underset{\substack{\uparrow \\ \text{extended pos. part}}}{\widehat{M^\alpha_+}}$ operator valued wt

Ex. 2.5

- $G \overset{\alpha}{\curvearrowright} L^\infty G$ translation.

  $$T_\alpha(f) = \int_G f(t)\,dt\, 1$$

  $$\mathfrak{m}^+_{T_\alpha} = L^1 G \cap L^\infty G_+ \underset{\text{dense}}{\subset} L^\infty G_+$$

- $G \overset{\alpha}{\curvearrowright} M$ integrable

  $\triangleright$ ${}^\forall e \in M^\alpha$ proj. $G \overset{\alpha^e}{\curvearrowright} eMe$ integrable

  $\triangleright$ ${}^\forall G \overset{\beta}{\curvearrowright} \cancel{M}\, N$ action. $G \overset{\alpha\otimes\beta}{\curvearrowright} M \otimes N$ integrable

- G abelian. & $G \overset{\alpha}{\curvearrowright} M$ action

  $\rightsquigarrow \hat{G} \overset{\hat{\alpha}}{\curvearrowright} M \rtimes_\alpha G$ integrable

  $$\begin{array}{c} M \rtimes_\alpha G \\ \cup \\ \mathbb{C} \otimes LG \simeq L^\infty \hat{G} \overset{\hat{\alpha}}{\curvearrowleft} \underset{\text{transl.}}{\hat{G}} \end{array}$$

Thm. 2.6 (Paschke)

$G \overset{\alpha}{\curvearrowright} M$ integrable action

$M \subset B(H)$ ← std. Hilb.

Let $U_\alpha(t)$ the implementing unitary of α_t

Then

$$\exists!\ \rho : M \rtimes_\alpha G \longrightarrow M \vee \{U_\alpha(t)\}_{t\in G}''$$

$$\pi_\alpha(x) \longmapsto x$$

$$\lambda^\alpha(t) \longmapsto U_\alpha(t)$$

normal surjective $*$-homo.

proof

Let $\varphi = \omega_{\xi_\varphi}$ f.n. state on M.

$\xi_\varphi \in P$. cyc & sep.

$\psi := \varphi \circ \int_G \alpha_t^{-1}(\cdot)\,dt \in W(M)$

α-inv. wt.

Set

$$V : H_\psi \longrightarrow H_\varphi \otimes L^2G$$

$$\Lambda_\psi(x) \longmapsto \left(V\Lambda_\psi(x)\right)(t) = \alpha_t^{-1}(x)\xi_\varphi, \quad t \in G$$

Then $V^*V = 1$ & $VV^* \in (M \rtimes_\alpha G)'$

$Vx = \pi_\alpha(x) V$

$VU^\alpha(t) = \lambda^\alpha(t) V$

$\left(\text{Note } U^\alpha(t)\Lambda_\psi(x) = \Lambda_\psi(\alpha_t(x))\right).$

$\rightsquigarrow \rho(x) = V^* x V \quad x \in M \rtimes_\alpha G$

works.

Dominant wt.

Defn. 2.7

$\varphi \in W(M)$ is

(1) of infinite multiplicity.

$\overset{\text{defn}}{\Longleftrightarrow}$ M_φ is properly infinite

$\left(\Longleftrightarrow \ M_\varphi \underset{\mathrm{vN}}{\cong} M_\varphi \otimes B(\ell^2)\right)$

(2) integrable

$\overset{\text{defn}}{\Longleftrightarrow}$ $\mathbb{R} \overset{\sigma^\varphi}{\curvearrowright} M$ integrable

(3) dominant

$\overset{\text{defn}}{\Longleftrightarrow}$

- φ inf. mult
- $^\forall \lambda > 0.\ ^\exists u \in M^u$ s.t. $\lambda\varphi = u\varphi u^*$

Ex. 2.8

$h := e^x \in C(\mathbb{R})_+$

Let $\omega := \mathrm{Tr}\, h \in W(B(L^2\mathbb{R}))$

$\rightsquigarrow$ $^\forall \lambda > 0.\ \lambda\omega \sim \omega$

Indeed, for $t \in \mathbb{R}$

$$\lambda(t)\,\omega\,\lambda(t)^* = \mathrm{Tr}\,\lambda(t) h \lambda(t)^*$$
$$= e^{-t}\,\mathrm{Tr}\, h$$
$$= e^{-t}\,\omega$$

Thus

$\varphi = \varphi_0 \otimes \omega \quad \in W(M \otimes B(L^2\mathbb{R}))$

is dominant if φ_0 inf. mult.

Lem. 2.9

M : prop. inf.

$Q \cong B(\ell^2)$.

$\exists \pi : M \longrightarrow M \otimes Q$ isomo.

s.t. $\forall \varphi \in W(M)$ inf

$\pi(\varphi) \sim \varphi \otimes \mathrm{Tr}_Q$

proof

Take $\underset{B(\ell^2)}{\overset{\cong}{Q_0}} \subset M$ s.t. $Q_0' \cap M$ prop. inf

e_{ij} : matrix unit.

$= V_i V_j^*$. $\{V_i\}_i$ ortho. isom.

$\sum V_i V_i^* = 1$.

Set

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M \otimes Q \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & \sum_{i,j} V_i^* x V_j \otimes e_{ij} \end{array} \quad \text{*-isomo.}$$

Take $\varphi \in W(M)$ inf. mult.

$\underset{B(\ell^2)}{\overset{\cong}{Q_1}} \subset M_\varphi$ & $Q_1' \cap M_\varphi$ prop. inf.

$M_\varphi \ni f_{ij} = W_i W_j^*$: mat. unit.

$$\begin{array}{ccc} M & \xrightarrow{\pi_W} & M \otimes Q \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \longmapsto & \sum W_i^* x W_j \cdot e_{ij} \end{array} \quad \text{isomo.}$$

Set $U = \sum_i W_i V_j^* \otimes e_{ij} \in M^u$

~~$\rightsquigarrow U e_{ij} U^* = f_{ij}$~~

$U \pi(x) U^* = \sum_{j,k} W_i^* V_j \cancel{V_j^*} x V_k \cancel{V_k^*} W_\ell \otimes e_{i\ell}$

$= \pi_W(x)$.

$U \pi(\varphi) U^* = \pi_W(\varphi) \underset{\uparrow}{=} \varphi \otimes \mathrm{Tr}$.

$(\varphi \otimes \mathrm{Tr}) \circ \pi_W(x) = \sum \varphi(W_i^* x W_i) = \varphi(x)$ ∎

Thm 2.10

A dominant wt is unique up to ~~conj~~ unitary conj. i.e. if $\varphi, \psi \in W(M)$ are dominant, then $\exists u \in M^u$ s.t.

$$u \varphi u^* = \psi$$

Proof

$$M \xrightarrow{\pi} M \otimes B(L^2\mathbb{R})$$

$$\varphi \longmapsto \pi(\varphi) \backsim \varphi \otimes \mathrm{Tr} \qquad \forall \varphi \in W(M)$$

Thus suffices to show

$$\varphi \otimes \mathrm{Tr} \sim \psi \otimes \mathrm{Tr} \quad \text{in } M \otimes B(L^2\mathbb{R})$$

$$\text{①}\ \backsim \qquad\qquad \backsim\ \text{③}$$

$$\varphi \otimes \omega \overset{②}{\sim} \psi \otimes \omega \qquad \omega = \mathrm{Tr}\,h$$

(Connes)

On ②,

Fourier transf.

$$\omega = \mathrm{Tr}\,h \sim \mathrm{Tr}\,k$$

$$k^{it} = \lambda(t).$$

Now let $V \in M \otimes L^\infty\mathbb{R}$

$$V(t) := [D\psi : D\varphi]_t$$

Then

$$\sigma_t^{\varphi \otimes \mathrm{Tr}k}(V)_{(s)} = (\sigma_t^\varphi \otimes \mathrm{Ad}\,\lambda(t))(V)_{(s)}$$

$$= \sigma_t^\varphi(V(-t+s))$$

$$= \sigma_t^\varphi(V(-t)\,\sigma_{-t}^\varphi(V(s)))$$

$$= \sigma_t^\varphi(V(-t))\,V(s)$$

$$= V(t)^*\,\underbrace{V(t)\,\sigma_t^\varphi(V(-t))}_{=1}\,V(s)$$

$$\rightsquigarrow \sigma_t^{\varphi \otimes \mathrm{Tr}k}(V) = (V(t)^* \otimes 1)\,V$$

coboundary!

$$[D\, V(\varphi\circ \mathrm{Tr}_k)V^* : D\varphi\circ \mathrm{Tr}_k]_t$$

$$= V\, \sigma_t^{\varphi\circ \mathrm{Tr}_k}(V^*)$$

$$= V(t)\otimes 1$$

$$= [D\psi\otimes \mathrm{Tr}_k : D\varphi\cdot \mathrm{Tr}_k]_t$$

Thus $V(\varphi\circ \mathrm{Tr}_k)V^* = \psi\otimes \mathrm{Tr}_k$

② $\blacksquare$

On ①. Set

$$G := \{ (t, u) \in \mathbb{R} \times \mathcal{U}(M) \mid u\varphi u^* = e^t \varphi \}$$

$$\Updownarrow$$

$$u\sigma_s^{\varphi}(u^*) = e^{its} \ \forall s$$

$\subset \mathbb{R} \times \mathcal{U}(M)$

closed subgrp $\to$ Polish

$G \xrightarrow{\mathrm{Pr}_1} \mathbb{R}$ surjective (& dominant) cont. grp homo

$\to$ open mapping

$G \xleftarrow{\exists S} \mathbb{R}$ Borel section

$u_t := \mathrm{Pr}_2 \circ S(t)$

$\rightsquigarrow \sigma_s^{\varphi}(u_t) = e^{-its} u_t \quad \forall s,\ \forall t \in \mathbb{R}$

Borel wrt $t \in \mathbb{R}$

Now let

$W \in M \otimes L^\infty \mathbb{R}$

$W(t) := u_t$

Then

$$\sigma_s^{\varphi \otimes \mathrm{Tr}}(W) = (1 \otimes e^{-is\cdot})\, W$$

$$\to [D\, W(\varphi \otimes \mathrm{Tr}) W^* : D(\varphi \otimes \mathrm{Tr})]_s$$

$$= W\, \sigma_t^{\varphi \otimes \mathrm{Tr}}(W^*)$$

$$= 1 \otimes e^{is\cdot}$$

$$= [D(\varphi \otimes \mathrm{Tr}\,h) : D(\varphi \otimes \mathrm{Tr})]_s$$

$$\rightsquigarrow \quad W(\varphi \otimes \mathrm{Tr}) W^* = \varphi \otimes \mathrm{Tr}\,h$$

∎

☆ dominant $\Rightarrow$ integrable.

φ : dominant

$\pi(\varphi) \sim \varphi \otimes \mathrm{Tr}$

$\quad \int$

$\quad \varphi \otimes \mathrm{Tr}\,h$

$\sigma_t^\varphi \otimes \underline{\underline{\mathrm{Ad}\,\lambda(t)}} \curvearrowright B(L^2\mathbb{R})$

$\cup$

$L^\infty \mathbb{R}$ integrable.

☆ $\overset{\text{type III}}{\downarrow}$

$M = N \rtimes_\theta \mathbb{R}$ cont. dec.

$\tau \cdot \theta_s = e^{-s} \tau$

$\varphi = \hat{\tau}$ is dominant.

Indeed,

$$\sigma_t^\varphi(\lambda^\theta(s)) = \hat{\theta}_t(\lambda^\theta(s))$$

$$= e^{-ist} \lambda^\theta(s)$$

$$[D \lambda^\theta(s) \varphi \lambda^\theta(s)^* : D\varphi]_t$$

$$= \lambda^\theta(s) \sigma_t^\varphi(\lambda^\theta(s)^*)$$

$$= e^{ist}$$

$$\rightsquigarrow \lambda^\theta(s) \varphi \lambda^\theta(s)^* = e^{s} \varphi.$$

$M_\varphi = N$ type II_∞

In particular, if M type III_1 factor,

M_φ is a II_∞ - factor.

Thm. 2.11

M : III_1-factor.

$\varphi \in W(M)$

Then φ dominant

$\Leftrightarrow$ φ integrable & inf. mult.

☆ In general.

integrable wt $<$ dom. wt.

proof

$\Rightarrow$ OK

$\Leftarrow$ Let φ, ψ : integrable & inf. mult.

$\chi := \begin{bmatrix} \varphi & \\ & \psi \end{bmatrix}$

$= e_{11}^* \otimes \varphi + e_{22}^* \otimes \psi$

$\in W(\mathbb{M}_2(M))$.

$N := \mathbb{M}_2 \otimes M$

Then χ is integrable.

$\left(\sigma_t^\chi(\ ) = \begin{pmatrix} \sigma_t^\varphi(\cdot) & * \\ * & \sigma_t^\psi(\cdot) \end{pmatrix}. \right)$

By Thm 2.6,

$\exists \rho : N \rtimes_{\sigma^\chi} \mathbb{R} \twoheadrightarrow N \vee \{U_{\sigma^\chi}(t)\}_t''$

$\uparrow$ II_∞-factor

(since N III_1 factor.)

Thus ρ is faithful.

$\longrightarrow N \vee \{U_{\sigma^\chi}(t)\}_t''$ II_∞-factor.

$\| $

$J_N N_\chi' J_N$.

☆ $G \overset{\alpha}{\curvearrowright} M \subset B(H)$ std.

$M^{\alpha} = M \cap \{U_\alpha(t)\}_t'$

$M^{\alpha\prime} = M' \vee \{U_\alpha(t)\}_t''$

$J_M M^{\alpha\prime} J_M = M \vee \{U_\alpha(t)\}_t''$

Hence N_x is type II factor.

$e_{11} N_x e_{11} = e_{11} \otimes M_\varphi$ $\quad \mathrm{II}_\infty$-factor.

$e_{22} N_x e_{22} = e_{22} \otimes M_\psi$ $\quad$ — // —

$\rightsquigarrow N_x$ II_∞ - factor.

& e_{11}, e_{22} are infinite

$\rightsquigarrow e_{11} \sim e_{22}$ in N

Then $\exists u \in N$

$e_{11} = u^* u$.

$e_{22} = u u^*$ $\quad \rightsquigarrow \quad u = \begin{bmatrix} 0 & 0 \\ v & 0 \end{bmatrix}$

$v \in M^u$

$$u = \sigma_t^x(u) = \begin{bmatrix} 0 & 0 \\ [D\psi : D\varphi]_t \, \sigma_t^\varphi(v) & 0 \end{bmatrix}$$

$\rightsquigarrow [D\psi : D\varphi]_t = v \, \sigma_t^\varphi(v^*)$

i.e. $\psi = v \varphi v^*$.

▨

Lem. 2.12

M : III_1-factor.

$\varphi \in W(M)$ inf mult.

Then $\forall \delta > 0$ $\exists h \in M_\varphi$

s.t.

- $e^{-\delta} 1_M \leq h \leq e^{\delta} 1_M$
- φ_h is dominant

proof

Recall $\lambda(t) = k^{it} \curvearrowright L^2\mathbb{R}$

$$k = \int_0^\infty y \, dE(\lambda)$$

$$p := E[\exp(-\delta), \exp(\delta)]$$

(inf. rank)

Now,

$$M \xrightarrow[\sim]{\pi} M \otimes B(\ell^2) \cong M \otimes B(L^2\mathbb{R})_p$$

$$\varphi \longmapsto \pi(\varphi)$$

$\quad\quad\quad\quad\quad\quad \wr$ unitary

$$\varphi \otimes \mathrm{Tr} \underset{\text{conj.}}{\approx} \varphi \otimes \mathrm{Tr}$$

Let $h := 1 \otimes k p$

Then

$$(\varphi \otimes \mathrm{Tr})_h = \varphi \otimes (\mathrm{Tr}\, k)_p$$

~~prep.~~ inf. mult. $\quad$ integrable. wt. $\mathrm{Ad}\,\lambda(t)p$

$\downarrow$ by Thm. 2.11

dominant

∎

Section 3

Connes' bicentralizer problem.

§3.1 Introduction

Defn. 3.1

M: vN alg.

$\varphi \in M_*$ faithful state.

- $AC(\varphi) := \{ (x_n) \in \ell^\infty(\mathbb{N}, M) \mid \lim_{n\to\infty} \| x_n\varphi - \varphi x_n \| = 0 \}$

  asymptotic centralizer

- $AB(\varphi) := \{ a \in M \mid \text{strong} \lim_{n\to\infty} (a x_n - x_n a) = 0 \text{ for } \forall (x_n) \in AC(\varphi) \}$

  asymptotic bicentralizer

☆

- $AC(\varphi) \subset \ell^\infty(\mathbb{N}, M)$ unital C^*-subalg.

  $\Downarrow$

  $1 = (1)_{n=1}^\infty$

- $AB(\varphi) \subset M_\varphi' \cap M$.

  $\uparrow$ vN subalg.

  $\uparrow$ because $M_\varphi \hookrightarrow AC(\varphi)$ as a const. seq.

Conjecture 3.2 (Connes)

For all III_1-factor M,

$$AB(\varphi) = \mathbb{C} \quad \text{for all } \varphi \in M_*^+ \text{ faithful state}$$

"some" is sufficient.

(by ~~Thm~~ Lem 3.8)

Ex. 3.3

$$R_\infty := \bigotimes_1^\infty \left(M_3(\mathbb{C}), \frac{1}{1+\lambda+\mu} \mathrm{Tr}\left(\begin{pmatrix} 1 & & \\ & \lambda & \\ & & \mu \end{pmatrix} \cdot \right) \right)''$$

$$\log \lambda / \log \mu \notin \mathbb{Q}$$

$\rightarrow$ type III_1 factor.

$$\varphi := \bigotimes_1^\infty \frac{1}{1+\lambda+\mu} \mathrm{Tr}\left(\begin{pmatrix} 1 & & \\ & \lambda & \\ & & \mu \end{pmatrix} \cdot \right)$$

$\rightsquigarrow (R_\infty)_\varphi' \cap R_\infty = \mathbb{C}$.

Hence $AB(\varphi) = \mathbb{C}$.

$$\Downarrow$$

$AB(\psi) = \mathbb{C}$ for $\forall \psi$ faithful state.

by

Thm. 3.4 (Connes)

M : injective III_1 - factor.

If $AB(\varphi) = \mathbb{C}$ for some φ,

then $M \cong R_\infty$

Sketch

$$AB(\varphi) = \mathbb{C}$$

$$\Downarrow$$

$\sigma_T^\varphi \in \overline{\mathrm{Int}}(M)$ $\forall T \in \mathbb{R}$

Connes classification of — Masuda-T. — // —

$\mathbb{T} \overset{\theta}{\curvearrowright} M \rtimes_{\sigma_T^\varphi} \mathbb{Z} \cong R_\lambda$ Powers factor. model action.

$\mathbb{R} \overset{\theta}{\curvearrowright} M \rtimes_{\sigma^\varphi} \mathbb{R} \cong R_{0,1}$ unique II_∞ factor. Rohlin flow.

Thm 3.5 (Haagerup)

M injective III_1-factor

$\Rightarrow AB(\varphi) = \mathbb{C}$

Idea.

ψ : dominant wt on M.

$M = M_\psi \rtimes_\theta \mathbb{R}$ cont. dec.

Connes–Takesaki
wt

$M_\psi' \cap M = \mathbb{C}$.

If ψ "were" state,

then "$AB(\psi) \subset M_\psi' \cap M = \mathbb{C}$" !

Want to replace ψ with φ !
$\uparrow$
state

By Lem 2.12, for $\varphi \otimes \mathrm{Tr}$ (inf mult) & $\delta > 0$

$\exists h \in M_\varphi \otimes B(\ell^2)$ s.t.

$$\begin{cases} \cdot\ e^{-\delta} \le h \le e^{\delta} \\ \cdot\ (\varphi \circ \mathrm{Tr})h \text{ is dominant.} \end{cases}$$

&

$\exists (a_n) \in \ell^\infty(\mathbb{N}, M)$

s.t.

$$\begin{cases} \cdot\ \sum a_i^* a_i = 1 \\ \cdot\ \mathrm{Sp}_{\sigma^\varphi}(a_i) \subset [-\delta, \delta] \\ \cdot\ \sum_{i=1}^{\infty} \|a_i x - x a_i\|_\varphi^2 \ge \frac{1}{2} \|(x - \varphi(x)1\|_\varphi^2 \end{cases}$$

a_i & φ are almost commuting

So, a_i's are "in $AC(\varphi)$" !

If $x \in AB(\varphi)$ then $x \approx \varphi(x)1$. ∎

§ 3.2 Basic Properties of $AB(\varphi)$

M : vN alg

φ : f.n. state on M.

Lem. 3.6 For $a \in M$, let

$$\boxed{\begin{gathered} C_\varphi(a,\delta) := \overline{\mathrm{co}}^{w}\{uau^* \mid u \in M^u,\ \|u\varphi - \varphi u\| \le \delta\}. \\ \text{Then} \\ a \in AB(\varphi) \iff \bigcap_{\delta>0} C_\varphi(a,\delta) = \{a\} \end{gathered}}$$

proof

Recall $AC(\varphi)$ is a C^*-alg

$\leadsto$ spanned by unitary seq's.

Hence

$a \in AB(\varphi)$

$\iff \lim_{n\to\infty} \|\cancel{au_n}\ \cancel{a} - au_n\|_\varphi = 0 \quad \forall (u_n) \in AC(\varphi)$

$\iff \lim_{n\to\infty} \|a - u_n^* a u_n\|_\varphi = 0 \quad$ —〃—

Put

$$\varepsilon(a,\delta) := \bigvee^{\sup} \{\|u^* a u - a\|_\varphi \mid u \in M^u,\ \|u\varphi - \varphi u\| \le \delta\}$$

Then

$a \in AB(\varphi) \iff \varepsilon(a,\delta) \xrightarrow[\delta\to 0]{} 0$ (Easy)

Let $x \in \mathrm{co}\{u^* a u \mid u \in \mathcal{M}^u,\ \|u\varphi - \varphi u\| \leq \delta\}$

$\rightsquigarrow \|x - a\|_\varphi \leq \varepsilon(a, \delta)$

$\rightsquigarrow \|x - a\|_\varphi \leq \varepsilon(a, \delta) \quad \forall x \in C_\varphi(a, \delta)$

($\uparrow$ lower semi-cont. of $y \mapsto \|y\|_\varphi$)

Hence. $a \in AB(\varphi) \Rightarrow \bigcap_{\delta > 0} C_\varphi(a, \delta) = \{a\}$.

Suppose $a \notin AB(\varphi)$

Then $\exists\, (u_n) \in AC(\varphi)^u$ s.t.

$\exists\, \varepsilon > 0$

$\|u_n^* a u_n - a\|_\varphi \geq \varepsilon. \ \forall n.$

$u_n^* a u_n \xrightarrow[\text{cluster}]{} b \in \bigcap_{\delta > 0} C_\varphi(a, \delta).$

$$\begin{aligned}
\|u_n^* a u_n\|_\varphi^2 &= \varphi(u_n^* a^* a \underline{u_n}) \\
&= u_n\varphi\,(u_n^* a^* a) \\
&\sim \varphi u_n\,(u_n^* a^* a) \\
&= \varphi(a^* a) \\
&= \|a\|_\varphi^2 .
\end{aligned}$$

Then

$$\begin{aligned}
2\,\mathrm{Re}\,\varphi(a^* u_n^* a u_n) &= \|a\|_\varphi^2 + \|u_n^* a u_n\|_\varphi^2 \\
&\quad - \|u_n^* a u_n - a\|_\varphi^2 \\
&\leq \|a\|_\varphi^2 + \|u_n^* a u_n\|_\varphi^2 \\
&\quad - \varepsilon^2
\end{aligned}$$

$\rightsquigarrow 2\,\mathrm{Re}\,\varphi(a^* b) \leq 2\|a\|_\varphi^2 - \varepsilon^2$

$\rightsquigarrow a \neq b$

Prop. 3.7

(1) $AB(\varphi) \subset M$ vN subalg.

(2) $AB(\varphi) = \mathbb{C}$

$\iff \forall a \in M.\ \forall \delta > 0.$

$\overline{\mathrm{co}}^{w} \{ u^* a u \mid u \in \mathcal{M}^u,\ \| u\varphi - \varphi u \| \leq \delta \}$

$\cap\ \mathbb{C} 1_M \neq \emptyset$

<u>proof</u>

(1) $a \in AB(\varphi)$

$\iff \bigcap_{\delta > 0} C_\varphi(a, \delta) = \{a\}$

$\updownarrow\ *$

$\iff \bigcap_{\delta > 0} C_\varphi(a^*, \delta) = \{a^*\}$

$\iff a^* \in AB(\varphi).$

(2) $a \in \overline{AB(\varphi)}^{\text{strong}}.$

$\varepsilon > 0.$ Take $b \in AB(\varphi)$ s.t.

$\| a - b \|_\varphi < \varepsilon.$

For $(u_n) \in AC(\varphi)^u$,

$\| u_n^* a u_n - a \|_\varphi$

$\leq \| u_n^* (a-b) u_n \|_\varphi + \| u_n^* b u_n - b \|_\varphi$

$+ \| b - a \|_\varphi$

$\downarrow 0$

$\overline{\lim_{n \to \infty}} \| u_n^* a u_n - a \|_\varphi \leq 2 \| a - b \|_\varphi < 2\varepsilon.$

$\square$

(2)

$\Leftarrow$. $C_\varphi(a,\delta) \cap \mathbb{C}1 \neq \emptyset$.

$\uparrow$ bdd & closed.

decreasing as $\delta \to 0$.

$\rightsquigarrow \bigcap_{\delta>0} C_\varphi(a,\delta) \cap \mathbb{C}1_M \neq \emptyset$

If $a \in AB(\varphi)$, then

$\bigcap_{\delta>0} C_\varphi(a,\delta) = \{a\}$ by

Thus $a \in \mathbb{C}1_M$.

$\Rightarrow$. $C_\varphi(a) := \bigcap_{\delta>0} C_\varphi(a,\delta)$ σ-w cpct.

$$\begin{array}{ccc} M & \longrightarrow & M\xi_\varphi \\ \Psi & & \Psi \\ x & \longmapsto & x\xi_\varphi \end{array}$$

σ-w to weak cont.

$\rightsquigarrow C_\varphi(a)\xi_\varphi \subset H_\varphi$ is weakly cpct convex.

$\rightsquigarrow$ norm closed.

Let $b \in C_\varphi(a)$ minimal dist. pt.

i.e. $\|b\xi_\varphi\| \leq \|x\xi_\varphi\| \quad \forall x \in C_\varphi(a)$

We will show $b \in AB(\varphi)$.

☆ $a' \in C_\varphi(a, \delta) \Longrightarrow C_\varphi(a', \delta) \subset C_\varphi(a, 2\delta)$.

(use $\|u\varphi u^* - \varphi\| \le \delta$, $\|v\varphi v^* - \varphi\| \le \delta$ $\Rightarrow \|uv\varphi v^* u^* - \varphi\| \le 2\delta$)

Hence

$$C_\varphi(b) = \bigcap_\delta C_\varphi(b, \delta)$$

$$\subset \bigcap_\delta C_\varphi(a, 2\delta)$$

$$= C_\varphi(a).$$

If $\|u\varphi - \varphi u\| \le \delta$, $u \in \mathcal{M}^u$

$$\|u^* b u\|_\varphi^2 = \varphi(u^* b^* b u)$$

$$= u\varphi(u^* b^* b)$$

$$= (u\varphi - \varphi u)(u^* b^* b) + \varphi(b^* b)$$

$$\le \|b\|_\varphi^2 + \underbrace{\|u\varphi - \varphi u\| \, \|b\|_\infty^2}_{\le \delta \|b\|_\infty^2}$$

$\to$ $x \in C_\varphi(b, \delta)$

$$\|x\|_\varphi^2 \le \|b\|_\varphi^2 + \delta \|b\|^2$$

$\leadsto$ $x \in C_\varphi(b) \subseteq C_\varphi(a)$

$$\Rightarrow \|x\|_\varphi^2 \le \|b\|_\varphi^2 \le \|x\|_\varphi^2$$

Hence $x = b$

i.e. $C_\varphi(b) = \{b\}$

Thus $b \in AB(\varphi) = \mathbb{C}$.

$$\underset{\substack{\psi \\ b}}{C_\varphi(a)} \cap \mathbb{C}1 \neq \emptyset.$$

Lem. 3.8 M type III_1 factor

If $AB(\varphi) = \mathbb{C}$ for some f.n. state φ

then $AB(\psi) = \mathbb{C}$ for $\forall$ f.n. state ψ

proof

Let $\delta > 0$ & $a \in M$.

By Connes – Størmer transitivity,

$\exists u \in M^u$ s.t.

$$\|\psi - u\varphi u^*\| < \delta.$$

Clearly $AB(u\varphi u^*) = \mathbb{C}$.

Hence

$$C_{u\varphi u^*}(a, \delta) \cap \mathbb{C} \neq \emptyset.$$

If $v \in M^u$ satisfies

$$\|v(u\varphi u^*) - (u\varphi u^*)v\| \leq \delta$$

then $\|v\psi - \psi v\| \leqq 3\delta$.

Hence.

$$C_{u\varphi u^*}(a, \delta) \subset C_\psi(a, 3\delta)$$

$\leadsto$ $C_\psi(a, 3\delta) \cap \mathbb{C}1 \neq \emptyset$.

§ 3.3 Main theorem.

Thm. 3.9 (Haagerup '87)

M : III_1-factor.

T.F.A.E.

(1) $\forall \psi \in W(M)$ dominant

$$\overline{\mathrm{co}}^{w}\{u x u^* \mid u \in M_\psi{}^u\} \cap \mathbb{C}1_M \neq \emptyset$$

for all $x \in M$

(2) $\forall \varphi \in M_*^+$ faithful state.

$$AB(\varphi) = \mathbb{C}$$

(3) $\exists \varphi \in M_*^+$ faithful state

s.t. $M_\varphi' \cap M = \mathbb{C}$

Cor. 3.10 (= Thm 3.5)

M injective III_1-factor

$\Longrightarrow$ Connes' bicentralizer problem ~~has~~ the affirmative~~ly~~ answer.

In particular

$$M \simeq R_\infty$$

proof

$$\begin{array}{c} M \\ \cup \;\downarrow E \\ M_\psi \end{array} \qquad E(x) = \int_{-\infty}^{\infty} \sigma_t^\psi(x)\, \underset{\text{inv. prob. measure}}{dm(t)}$$

$\rightsquigarrow$ M_ψ is injective

$\rightsquigarrow$ (1) holds since $M_\psi' \cap M = \mathbb{C}$ & Schwartz property

$\boxed{\text{MW}}$

(3) $\Rightarrow$ (1)

$$M \cong M \otimes B(L^2\mathbb{R}) =: N$$

$$\varphi \otimes \mathrm{Tr}_k =: \psi$$

$$k^{it} = \lambda(t).$$

$$M_\varphi \otimes L\mathbb{R} \subset N_\psi$$

By Popa's result,

$$\exists A \subset M_\varphi \subset M$$

common MASA.

$\rightsquigarrow$ $A \otimes L\mathbb{R} =: B \subset N$ MASA

$$\overline{\mathrm{co}}^w \{uxu^* \mid u \in N_\psi^u\}$$

$$\cup$$

$$\overline{\mathrm{co}}^w \{uxu^* \mid u \in B^u\} \cap (B' \cap N) \neq \emptyset$$

$$B' \cap N = B \subset N_\psi$$

$$\ni y$$

$$\overline{\mathrm{co}}^w \{uyu^* \mid u \in N_\psi^u\} \cap Z(N_\psi) \neq \emptyset$$

$$Z(N_\psi) = \mathbb{C}$$

Dixmier

§3.4 Proof of Thm 3.9 (1) $\Rightarrow$ (2) part I

ψ : dominant wt

$\leadsto$ $M_\psi' \cap M = \mathbb{C}$

Want to replace ψ with a state!

Lem. 3.11 (Dominant $\to$ inf. mult.)

$$\boxed{\begin{array}{l} M \text{ w. Thm 3.9 (1).} \\ \varphi \in W(M) \quad \text{inf. mult.} \\ \text{Then } \forall x \in M \quad \forall \delta > 0 \\ \overline{\mathrm{co}}^w \{ uxu^* \mid u \in M^u,\ \mathrm{Sp}_\varphi(u) \subset [-\delta, \delta] \} \\ \qquad \cap\ \mathbb{C}1 \neq \emptyset \end{array}}$$

Proof

By Lem 2.12, $\exists h \in M_\varphi$ s.t.

- $e^{-\frac{\delta}{2}} \leq h \leq e^{\frac{\delta}{2}}$
- φ_h is dominant.

Thus

$$\overline{\mathrm{co}}^w \{ uxu^* \mid u \in M^u_{\varphi_h},\ \mathrm{Sp}\cancel{_{\varphi_h}(u)} \} \cap \mathbb{C}1 \neq \emptyset$$

If $x \in M_{\varphi_h}$, then

$$\sigma_t^\varphi(x) = h^{-it}\, \sigma_t^{\varphi_h}(x)\, h^{it}$$
$$= h^{-it} x h^{it}.$$

$$\|\sigma_z^\varphi(x)\| \leq \|h^{-iz}\|\, \|h^{iz}\|\ \|x\|$$
$$\leq e^{2 \cdot \frac{\delta}{2} |\mathrm{Im}\, z|}\ \|x\|.$$

Prop 1.14 implies $M \cancel{_{\varphi_h}} \cancel{x \in}$ $\mathrm{Sp}_{\sigma^\varphi}(x) \subset [-\delta, \delta]$

Let us apply Lem to $\varphi \otimes \mathrm{Tr} =: \psi$

$\psi \in$ state. $W(M \otimes B(\ell^2))$

$W(M \otimes B(\ell^2)) = \ddot{N}$

For $x \otimes 1 \in M \otimes B(\ell^2)$ & $\delta > 0$,

$\exists \lambda \in \mathbb{C}$

$$\lambda(1 \cdot 1) \in \overline{\mathrm{co}}^{w}\{u(x \cdot 1)u^* \mid u \in N^u, \ \mathrm{sp}_\psi(u) \subset [-\delta, \delta]\}.$$

$$\rightsquigarrow x \cdot 1 - \lambda(1 \cdot 1) \in \overline{\mathrm{co}}^{w}\{x \cdot 1 - u(x \cdot 1)u^* \mid \text{—''—}\} = \overline{\mathrm{co}}^{s}\{\quad\}$$

$\forall \zeta \in H \cdot \ell^2$

$$\|((x - \lambda) \cdot 1)\zeta\| \leq \sup_u \|((x \cdot 1) - u(x \cdot 1)u^*)\zeta\|$$

$$= \sup_u \|(u^*(x \cdot 1) - (x \cdot 1)u^*)\zeta\|$$

Let $\xi \in H$ cyc & sep unit vector.

$\zeta := \xi \otimes e_1$.

$\exists u \in N^u$ & $\mathrm{sp}_\psi(u) \subset [-\delta, \delta]$

s.t.

$$\|(x - \lambda)\xi\| \leq \sqrt{2}\, \|(u^*(x \cdot 1) - (x \cdot 1)u^*)(\xi \cdot e_1)\|$$

Let $u^* = (a_{ij})_{i,j=1}^{\infty}$, $a_{ij} \in M$.

then $\mathrm{Sp}_\varphi(a_{ij}) \subset [-\delta, \delta]$

&

$$\|(x - \lambda)\xi\|^2 \leq \sqrt{2} \sum_{i=1}^{\infty} \|(a_{i1}x - x a_{i1})\xi\|^2$$

Now. let $\mathring{x} := x - \varphi(x)1$. then

$\mathring{x}\xi \perp \mathbb{C}\xi$ $\quad \varphi = \langle \cdot\, \xi, \xi \rangle$

$$\|(x - \lambda)\xi\|^2 = \|\mathring{x} + (\varphi(x) - \lambda)\xi\|^2$$

$$\geq \|\mathring{x}\xi\|^2.$$

Lem. 3.12

M w. Thm 3.9 (1).

$\varphi \in M_*$: faithful state.

$x \in M$.

Then $\forall \delta > 0$, $\exists (a_i)_{i=1}^{\infty}$ in M.

s.t.

(1) $\quad Sp_{\sigma^\varphi}(a_i) \subset [-\delta, \delta]$ $\quad \forall i$

(2) $\quad \sum_{i=1}^{\infty} a_i^* a_i = 1$

(3) $\quad \sum_{i=1}^{\infty} \| x a_i - a_i x \|_\varphi^2 \geq \frac{1}{2} \| \mathring{x} \|_\varphi^2$

where $\mathring{x} := x - \varphi(x) 1$

σ^φで大体止まる
i.e. φと大体交換
$\xrightarrow{\delta \downarrow 0} AC(\varphi)$.

大体 a_iたちと交換

$\rightsquigarrow \mathring{x} \sim 0$

i.e. $x \sim \varphi(x)$
$\qquad\qquad\quad \cap$
$\qquad\qquad\quad \mathbb{C}$.

$\rightsquigarrow$ maybe $AB(\varphi) = \mathbb{C}$

☆ $Sp_{\sigma^\varphi}(a) \subset [-\delta, \delta]$

$\Longrightarrow \| a \xi_\varphi - \xi_\varphi a \| \leq \frac{\delta}{2} \| a \xi_\varphi \|$.

proof

$\Delta_\varphi = e^H = \int_{-\infty}^{\infty} e^{\lambda} \, dE(\lambda)$.

$\Delta_\varphi^{it} = \int_{-\infty}^{\infty} e^{i\lambda t} \, dE(\lambda)$.

$\rightsquigarrow a \xi_\varphi \in E[-\delta, \delta] H_\varphi$.

Then

$\| \Delta_\varphi^{1/2} a \xi_\varphi - a \xi_\varphi \|^2$

$= \int_{-\delta}^{\delta} | e^{\frac{\lambda}{2}} - 1 |^2 \, d\| E(\lambda) a \xi_\varphi \|^2$

$\leq (e^{\frac{\delta}{2}} - 1)^2 \| a \xi_\varphi \|^2$

$\leq \frac{\delta^2}{4} \| a \xi_\varphi \|^2$.

一方で

$$\xi_\varphi a = J a^* J \xi_\varphi$$
$$= J a^* \xi_\varphi$$
$$= J J \Delta_\varphi^{1/2} a \xi_\varphi$$
$$= \Delta_\varphi^{\frac{1}{2}} a \xi_\varphi \qquad ■$$

まとめると，

$\forall x \in M$，$\forall \delta > 0$．

$\exists (a_i)_{i=1}^\infty$ in M．

modular auto が消えた ⇝ Bimodule ${}_M H_M$ の幾何に持ち込む．

s.t.

$$\begin{cases} \cdot\ \|a_i \xi_\varphi - \xi_\varphi a_i\| \leq \delta \|a_i \xi_\varphi\| \quad \forall i & \text{—①} \\ \cdot\ \sum_{i=1}^\infty a_i^* a_i = 1 & \text{—②} \\ \cdot\ \sum_{i=1}^\infty \|a_i x - x a_i\|_\varphi^2 \geq \frac{1}{2} \|x \xi_\varphi\|_\varphi^2 & \end{cases}$$

次の目標： よい a_i を見つけて．$\sum_{i=1}^\infty$ を消す．

ここで

$$\left(a_i\, x\xi_\varphi - x\xi_\varphi a_i\right)_i$$

$$= \left((a_i x - x a_i)\xi_\varphi\right)_i - \left(x(a_i\xi_\varphi - \xi_\varphi a_i)\right)_i$$

$\in \ell^2(\mathbb{N}, H)$. by ① & ②

$$\rightsquigarrow \left(\sum_i \| a_i\, x\xi_\varphi - x\xi_\varphi a_i\|^2\right)^{\frac{1}{2}}$$

$$\geq \left(\sum_i \|a_i x - x a_i\|_\varphi^2\right)^{\frac{1}{2}} - \|x\|\,\delta\left(\sum_i \|a_i\xi_\varphi\|^2\right)^{\frac{1}{2}}$$

$\left(\sum_i \|a_i\xi_\varphi\|^2\right)^{\frac{1}{2}} = 1$ by ②.

$$\geq \frac{1}{\sqrt{2}}\|\mathring{x}\|_\varphi - \delta\|x\|.$$

x を $\mathring{x}$ に ~~変えてもOK~~

$\eta = \mathring{x}\xi_\varphi$ ~~として~~ Lem を示す.

を一般化して

Lem. 3.13

M Thm 3.9 (1)

- $\xi \in P$ cyc & sep unit
- $\eta = J\eta$, $\eta \perp \xi$, $\|\eta\| = 1$ ($\mathring{x}$ の代わり AB(φ) を想起)

$\forall \delta > 0$ $\exists a \in M\setminus\{0\}$

s.t.

- $\|a\xi - \xi a\|^2 < \delta\, \|a\eta - \eta a\|^2$ ← aとξは大体交換 AC(φ)と想起
- $\|a\xi\|^2 + \|a\eta\|^2 < 8\, \|a\eta - \eta a\|^2$ ← aとηはあまり交換しない.

proof

$\varphi := \langle\cdot\,\xi, \xi\rangle$ f.n.state on M

$\psi := \langle\cdot\,\eta, \eta\rangle$ state.

<u>Case 1</u>. $\exists K > 0$. s.t. $\psi \leq K\varphi$.

Then

$$x' : H_\varphi \to H_\varphi, \quad y\xi_\varphi \mapsto y\eta \qquad x' \in M', \quad \|x'\| \leq K^{\frac{1}{2}}$$

Let $x := Jx'J$

$$x\xi = Jx'J\xi = Jx'\xi = J\eta = \eta.$$

$$\varphi(x) = \langle x\xi, \xi\rangle = \langle \eta, \xi\rangle = 0.$$

Take a_i & $\delta_1 < \left(\frac{\delta}{8}\right)^{\frac{1}{2}}$

- $\| a_i \xi - \xi a_i \| \le \delta_1 \| a_i \xi \|$
- $\sum_i a_i^* a_i = 1$
- $\left(\sum_i \| a_i \eta - \eta a_i \|^2\right)^{\frac{1}{2}} \ge \cancel{\frac{1}{\sqrt{2}} - K^{\frac{1}{2}}}$

$$\frac{1}{\sqrt{2}} \| \underset{\underset{\eta}{\parallel}}{x\xi_\varphi} \| - \|x\| \delta_1$$

$$\ge \frac{1}{\sqrt{2}} - K^{\frac{1}{2}} \delta_1$$

δ_1 を $\frac{1}{\sqrt{2}} - K^{\frac{1}{2}} \delta_1 \ge \frac{7}{8\sqrt{2}}$

となるようにとっておく。

$\rightarrow$

$$8 \times \sum_i \| a_i \eta - \eta a_i \|^2 \ge 8 \times \frac{49}{64} \cdot \frac{1}{2} > 3$$

一方

$$\sum_i \left(\| a_i \xi \|^2 + \| a_i \eta \|^2 + 8\delta^{-1} \| a_i \xi - \xi a_i \|^2 \right)$$

$$\le 1 + 1 + \underbrace{8\delta^{-1} \cdot \delta_1^2}_{< 1}$$

$$< 3.$$

よってある i_0 において

$$\| a_{i_0} \xi \|^2 + \| a_{i_0} \eta \|^2 + 8\delta^{-1} \| a_{i_0} \xi - \xi a_{i_0} \|^2$$

$$< 8 \| a_{i_0} \eta - \eta a_{i_0} \|^2$$

とくに $a_{i_0} \ne 0$.

Case 1 OK

<u>Case 2</u> ψ is not dominated by $\mathbb{R}_+ \varphi$.

$\exists \underset{\neq 0}{p} \in M^P$ s.t. $\psi(p) > \frac{16}{8} \varphi(p)$.

pMp no atom

$\rightsquigarrow \{ \psi(q) \mid q \in pMp \text{ proj} \} = [0, \psi(p)]$

Take $q \in pMp$ proj s.t.

$$\psi(q) = \frac{1}{16} \overset{\text{psi}}{\psi}(p) \neq 0$$

Then $q, \ p - q \neq 0$

Take $v \in M$ s.t.

$$v^* v = p - q \ \& \ v v^* = q$$

Then

$$\|v\eta\|^2 = \psi(p-q) = \frac{15}{16}\psi(p)$$

$$\|\eta v\|^2 = \|J v^* \underbrace{J\eta}_{=\eta}\|^2 = \psi(q) = \frac{1}{16}\psi(p).$$

$$\rightsquigarrow \|v\eta - \eta v\| \geq \|v\eta\| - \|\eta v\|$$
$$\geq \left(\frac{\sqrt{15}}{4} - \frac{1}{4}\right) \psi(p)^{\frac{1}{2}}$$
$$\geq \frac{1}{2} \psi(p)^{\frac{1}{2}} \quad \text{—①}$$

$$\|v\xi\|^2 + \|v\eta\|^2 = \varphi(p-q) + \psi(p-q)$$
$$< \left(\frac{8}{16} + 1\right) \psi(p)$$
$$< 2\psi(p)$$
$$\leq 8 \|v\eta - \eta v\|^2 \quad \text{by ①}.$$

$$\|v\xi - \xi v\|^2 \leq 2\|v\xi\|^2 + 2\underbrace{\|\xi v\|^2}_{=\|v^*\xi\|}$$

$$= 2\varphi(p-q) + 2\varphi(q)$$

$$\approx 2\varphi(p)$$

$$< \frac{\delta}{8}\,\psi(p)$$

$$\leq \frac{\delta}{2}\,\|v\eta - \eta v\|^2$$

Case 2 ∎

<u>次の目標</u>： a を self-adjoint にする．

$\downarrow$ Connes' joint distribution.

projection

Lem 3.14

M, ξ, η as before

$\forall \delta > 0 \quad \exists\, b \in M_{s.a.} \setminus \{0\}$

s.t.

- $\|b\xi\|^2 + \|b\eta\|^2 < 32 \|b\eta - \eta b\|^2$
- $\|b\xi - \xi b\|^2 < \delta \|b\eta - \eta b\|^2$

Proof

Take $a \in M_{sa} \setminus \{0\}$. s.t.

$$\begin{cases} \|a\xi\|^2 + \|a\eta\|^2 < 8 \|a\eta - \eta a\|^2 \\ \|a\xi - \xi a\|^2 < \frac{\delta}{4} \|a\eta - \eta a\|^2 \end{cases}$$

Let $a = \underset{M_{sa}.}{b_1} + i\, \underset{M_{sa}.}{b_2}$

WMA $b_1 \neq 0 \neq b_2$.

$$\begin{aligned} \|a^*\xi\| &= \|Ja^*J\xi\| \\ &= \|\xi a\| \\ &\leq \|\xi a - a\xi\| + \|a\xi\| \\ &\leq \left(\left(\frac{\delta}{4}\right)^{\frac{1}{2}} + 8^{\frac{1}{2}}\right) \|a\eta - \eta a\| \\ &< 4 \|a^*\eta - \eta a^*\|. \end{aligned}$$

$\|a\eta - \eta a\| \overset{J}{=} \|\eta a^* - a^*\eta\|$

$J(\xi y) = y^* J\xi$

$$\begin{aligned} \|a^*\eta\| &= \|\eta a\| \\ &\leq \|\eta a - a\eta\| + \|a\eta\| \\ &< (1 + 8^{\frac{1}{2}}) \|a\eta - \eta a\| \\ &< 4 \|a^*\eta - \eta a^*\|. \end{aligned}$$

$\|a^*\xi - \xi a^*\| = \|a\xi - \xi a\|.$

Thus

$$\|a^*\xi\|^2 + \|a^*\eta\|^2 + 32\delta^{-1}\|a^*\xi - \xi a^*\|^2$$

$$< \left(4^2 + 4^2 + 32\delta^{-1}\cdot\frac{\delta}{4}\right)\|a^*\eta - \eta a^*\|^2$$

$$= 40\|a^*\eta - \eta a^*\|^2$$

$$\|a\xi\|^2 + \|a\eta\|^2 + 32\delta^{-1}\|a\xi - \xi a\|^2$$

$$< \left(8 + 32\delta^{-1}\cdot\frac{\delta}{4}\right)\|a\eta - \eta a\|^2$$

$$= 16\|a\eta - \eta a\|^2$$

中線定理.

$$\|a\xi\|^2 + \|a^*\xi\|^2 = 2\|b_1\xi\|^2 + 2\|b_2\xi\|^2$$

etc.

より. 2で割ると.

$$\|b_1\xi\|^2 + \|b_2\xi\|^2$$

$$+ \|b_1\eta\|^2 + \|b_2\eta\|^2$$

$$+ 32\delta^{-1}\|b_1\xi - \xi b_1\|^2 + 32\delta^{-1}\|b_2\xi - \xi b_2\|^2$$

$$< 28\|b_1\eta - \eta b_1\|^2 + 28\|b_2\eta - \eta b_2\|^2$$

よって. $b = b_1$ or b_2 とすれば OK

次の目標 : b を射影にする.

§3.5 Connes' joint distribution

M : vN alg

$h, k \in M_{sa}$.

Let $X := Sp(h) \subset \mathbb{R}$

$Y := Sp(k) \subset \mathbb{R}$.

std. Hilb.

$$\begin{array}{ccc} C(X) \underset{alg}{\otimes} C(Y) & \longrightarrow & B(H) \\ \Psi & & \Psi \\ f \otimes g & \longmapsto & f(h)\, J g(k)^* J \end{array}$$

nuclear

$\leadsto$ extends to $*$-homo.

$\pi : C(X) \otimes_{min} C(Y) \longrightarrow B(H)$

$\|$

$C(X \times Y)$.

Let $\xi \in H$ unit vector.

$$\begin{array}{ccc} C(X \times Y) & \longrightarrow & \mathbb{C} \\ \Psi & & \Psi \\ a & \longmapsto & \langle \pi(a)\xi, \xi \rangle \end{array}$$

is a state

$\Updownarrow$

$\exists!\ \mu$: Borel prob. measure on $X \times Y$.
regular.

s.t.

$$\langle \pi(a)\xi, \xi \rangle = \int_{X \times Y} a(x, y)\, d\mu(x, y)$$

In particular,

$a = f \otimes g$

$$\leadsto \langle f(h)\xi g(k), \xi \rangle = \int_{X \times Y} f(x) g(y)\, d\mu(x, y)$$

$\|$

$\int_{\mathbb{R}} f(\lambda)\, dE(\lambda)$

regular

regular

This equality holds for f, g bdd Borel.

$\mu(\mathbb{R}^2 \setminus X \times Y) = 0$ として.

μ を $\mathbb{R}^2$ 上の measure に拡張.

Lem. 3.15

$$\boxed{\begin{array}{l} f, g : \text{bdd Borel on } \mathbb{R}. \\ \| f(h)\xi - \xi g(k) \|^2 = \displaystyle\int_{\mathbb{R}^2} |f(s) - g(t)|^2 \, d\mu(s,t) \end{array}}$$

proof

$$\text{LHS} = \langle (\bar{f} f)(h)\xi, \xi \rangle + \langle \xi \cancel{g\bar{g}(k), g} \; g\bar{g}(k), \xi \rangle$$

$$- 2\,\mathrm{Re} \langle f(h)\xi \bar{g}(k), \xi \rangle$$

$$= \int_{\mathbb{R}^2} \left(|f(s)|^2 + |g(t)|^2 - 2\,\mathrm{Re}\, f(s)\overline{g(t)} \right) d\mu(s,t)$$

$= \text{RHS}.$ ▨

Lem. 3.16

> $b \in M_{sa}$. $\zeta \in H_{sa}$ i.e. $J\zeta = \zeta$.
>
> $$b = \int_{-\infty}^{\infty} \lambda \, de(\lambda) \quad \text{(spectral res.)}$$
>
> Then
>
> $$\frac{1}{4}\|b\zeta - \zeta b\|^2 \leq \int_{-\infty}^{\infty} \|e(\lambda)\zeta - \zeta e(\lambda)\|^2 |\lambda| d\lambda$$
>
> $$\leq \|b\zeta\| \, \|b\zeta - \zeta b\|.$$

proof

μ : $h = k = b$ に対する joint distr.

$e(\lambda) = 1_{(-\infty, \lambda]}(b)$ より.

$$\|e(\lambda)\zeta - \zeta e(\lambda)\|^2 = \int_{\mathbb{R}^2} \underbrace{\left|1_{(-\infty,\lambda]}(s) - 1_{(-\infty,\lambda]}(t)\right|^2}_{= h(s,t,\lambda)} d\mu(s,t)$$

$$h(s,t,\lambda) = \begin{cases} 1 & \text{if } s \leq \lambda \lneq t \text{ or } t \leq \lambda \lneq s. \\ 0 & \text{otherwise.} \end{cases}$$

By Fubini,

$$\int_{-\infty}^{\infty} \|e(\lambda)\zeta - \zeta e(\lambda)\|^2 |\lambda| d\lambda$$

$$= \int_{\mathbb{R}^2} d\mu(s,t) \int_{-\infty}^{\infty} d\lambda \, h(s,t,\lambda) |\lambda|.$$

Claim.

$$\int_{-\infty}^{\infty} h(s,t,\lambda)|\lambda| d\lambda = \frac{1}{2}\left| t^2 \operatorname{sign} t - s^2 \operatorname{sign} s \right|$$

proof

Note $h(s,t,\lambda) = h(t,s,\lambda)$

WMA $s \leq t$.

$$\int_{-\infty}^{\infty} h(s,t,\lambda)|\lambda| d\lambda = \int_s^t |\lambda| d\lambda$$

$= \text{RHS}$ Claim 1. ∎

Claim 2.

$$\frac{1}{2}(s-t)^2 \underset{(a)}{\leq} |s^2 \operatorname{sign} s - t^2 \operatorname{sign} t| \underset{(b)}{\leq} |s-t|(|s|+|t|)$$

proof

When $st \geq 0$.

$$|s^2 \operatorname{sign} s - t^2 \operatorname{sign} t| = |s^2 - t^2|$$

$$= |s-t||s+t| \overset{(b)}{\leq} |s-t|(|s|+|t|)$$

$$\underset{(a)}{\geq} \frac{1}{2}(s-t)^2$$

When $st < 0$

$$|s^2 \operatorname{sign} s - t^2 \operatorname{sign} t| = (s^2+t^2) \overset{(b)}{\leqq} \underbrace{|s-t|(|s|+|t|)}_{=(|s|+|t|)(|s|+|t|)}$$

$$\underset{(a)}{\geq} \frac{1}{2}(s-t)^2$$

Claim 2 $\boxed{}$

Hence

$$\frac{1}{4}(s-t)^2 \leq \int_{-\infty}^{\infty} h(s,t,\lambda) \frac{\cancel{d\lambda}}{|\lambda|} d\lambda \leq \frac{1}{2} \cdot |s-t|(|s|+|t|)$$

$$\frac{1}{4}\int_{\mathbb{R}^2} (s-t)^2 \, d\mu(s,t) = \frac{1}{4}\|b\xi - \xi b\|^2$$

$$\frac{1}{2}\int_{\mathbb{R}^2} |s-t|(|s|+|t|) \, d\mu(s,t)$$

$$\leq \frac{1}{2}\left(\int_{\mathbb{R}^2} |s-t|^2 d\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \underbrace{(|s|+|t|)^2}_{\leq 2|s|^2+2|t|^2} d\mu\right)^{1/2}$$

$$= \frac{1}{2}\|b\xi - \xi b\| \underbrace{(2\|b\xi\|^2 + \underbrace{2\|\xi b\|^2}_{=\|b\xi\|^2})^{1/2}}_{=2\|b\xi\|}$$

$\boxed{}$

§3.6 Proof of Thm 3.9 (1)⇒(2) Part II.

Lem. 3.17

M : III_1-factor with Thm 3.9(1).

$\xi \in \mathcal{P}$ cyc & sep. $\|\xi\| = 1$

$\eta \in H$ $J\eta = \eta$, $\|\eta\| = 1$

$\eta \perp \xi$

$\forall \delta > 0$ $\exists p \in M^p$ non-zero.

s.t.

- $\|p\xi\|^2 + \|p\eta\|^2 < 2^7 \|p\eta - \eta p\|^2$
- $\|p\xi - \xi p\|^2 < \delta \|p\eta - \eta p\|^2$

proof

Let $\delta_1 > 0$.

Take $b \in M_{sa}$ as in Lem :

- $\|b\xi\|^2 + \|b\eta\|^2 < 32 \|b\eta - \eta b\|^2$
- $\|b\xi - \xi b\|^2 < \delta_1 \|b\eta - \eta b\|^2$

Let

$$b = \int_{-\infty}^{\infty} \lambda \, de_\lambda \quad \text{(spectral res.)}$$

Want to replace b with e_λ !

$$\int_{-\infty}^{\infty} \|e_\lambda \xi\|^2 |\lambda| \, d\lambda \quad \text{vs} \quad \|b\xi\|^2 \quad ?$$

but $\lambda \to +\infty \Rightarrow \|e_\lambda \xi\| \to \|\xi\|$

Thus $\int_{-\infty}^{\infty} \text{—}\,\text{''}\,\text{—} \, d\lambda = +\infty$.

So, we need to arrange e_λ as follows

$$f_\lambda := \begin{cases} e_\lambda & \text{if } \lambda < 0 \\ 1 - e_\lambda & \text{if } \lambda \geqq 0 \end{cases}$$

Note $e_\lambda = 1_{(-\infty, \lambda]}(b)$

$1 - e_\lambda = 1_{(\lambda, \infty)}(b)$

Then for $\xi \in H$,

$$\int_{-\infty}^{\infty} \|f_\lambda \xi\|^2 |\lambda|\, d\lambda$$

$$= \int_{-\infty}^{0} d\lambda\, |\lambda| \int_{\mathbb{R}^2} d\mu(s,t)\, \Big|\underbrace{1_{(-\infty, s]}(s)}_{\substack{\| \\ 1_{[s,\infty)}(\lambda)}}\Big|^2$$

$$+ \int_{0}^{\infty} d\lambda\, |\lambda| \int_{\mathbb{R}^2} d\mu(s,t)\, \Big|\underbrace{1_{(s,\infty)}(\rho)}_{\substack{\| \\ 1_{(-\infty,s)}(\lambda)}}\Big|^2$$

Fubini:

$$= \int_{\mathbb{R}^2} d\mu(s,t) \int_{-\infty}^{0} d\lambda\, |\lambda|\, \underbrace{1_{[s,\infty)}(\lambda)}_{=0 \text{ if } s \geq 0}$$

$$+ \int_{\mathbb{R}^2} d\mu(s,t) \int_{0}^{\infty} d\lambda\, |\lambda|\, \underbrace{1_{(-\infty,s)}(\lambda)}_{=0 \text{ if } s \leq 0}$$

$$= \int_{\mathbb{R}^2} d\mu(s,t) \Big(\underbrace{\int_{s}^{0} d\lambda\, |\lambda|}_{=\frac{s^2}{2}}\Big)\, 1_{(-\infty,0]}(s)$$

$$+ \int_{\mathbb{R}^2} d\mu(s,t) \Big(\underbrace{\int_{0}^{s} d\lambda\, |\lambda|}_{=\frac{s^2}{2}}\Big)\, 1_{(0,\infty)}(s)$$

$$= \int_{\mathbb{R}^2} d\mu(s,t)\, \frac{s^2}{2} \underbrace{\Big(1_{(-\infty,0]}(s) + \frac{s^2}{2} 1_{(0,\infty)}(s)\Big)}_{\substack{\| \\ 1}}$$

$$= \frac{1}{2} \cdot \|b\xi\|^2$$

よって.

$$\int_{-\infty}^{\infty} \|f_\lambda \xi\|^2 |\lambda|\, d\lambda = \frac{1}{2} \|b\xi\|^2 \quad \forall \xi \in H.$$

ここで

$$f_\lambda \xi - \xi f_\lambda = I(e_\lambda \xi - \xi e_\lambda) \quad \forall \lambda \in \mathbb{R}$$

に注意すると,

$$\int_{-\infty}^{\infty} \Big(\|f_\lambda \xi\|^2 + \|f_\lambda \eta\|^2 + \delta_1^{-\frac{1}{2}} \|f_\lambda \xi - \xi f_\lambda\|^2 \Big) |\lambda|\, d\lambda$$

$$\leq \frac{1}{2} \|b\xi\|^2 + \frac{1}{2} \|b\eta\|^2 + \delta_1^{-\frac{1}{2}} \|b\xi\|\, \|b\xi - \xi b\| \quad \text{by Lem}$$

$$< \frac{32}{2} \|b\eta - \eta b\|^2 + \underbrace{\|b\xi\|}_{< 32^{\frac{1}{2}} \|b\eta - \eta b\|} \|b\eta - \eta b\|.$$

$$< 32 \|b\eta - \eta b\|^2$$

$$\|b\xi\|^2 = \int_{-\infty}^{\infty} \lambda^2 \, d\|E(\lambda)\xi\|^2$$

$$= \int_{-\infty}^{0} + \int_{0}^{\infty}$$

$$\int_{-\infty}^{0} = \left[\lambda^2 \|E(\lambda)\xi\|^2\right]_{-\infty}^{0} - \int_{-\infty}^{0} \|E(\lambda)\xi\|^2 \, 2\lambda d\lambda$$

$$= \int_{-\infty}^{0} \|E(\lambda)\xi\|^2 |\lambda| \, d\lambda \times 2.$$

$$\int_0^{\infty} \lambda^2 \, d\|f_\lambda \xi\|^2 = -\int_0^{\infty} \|f_\lambda \xi\|^2 \, 2\lambda \, d\lambda$$

$$= -\int_0^{\infty} (1 - \|e_\lambda \xi\|^2) \, 2\lambda \, d\lambda$$

$$= \int_0^{\infty} \lambda^2 \, d(\quad)$$

$$\int_{-\infty}^{\infty} \|f_\lambda \xi\|^2 |\lambda| \, d\lambda$$

$$= \int_0^{\infty} d\lambda \, |\lambda| \int_{-\infty}^{\infty} \underset{\underset{1_{(0,\rho)}(\lambda)}{\parallel}}{1_{(\lambda,\infty)}(\rho)} \, d\|e_\rho \xi\|^2$$

$$= \int_{(0,\infty)} d\|e_\rho \xi\|^2 \cdot \underbrace{\int_0^{\rho} d\lambda \, \lambda}_{\frac{\rho^2}{2}}$$

$$= \frac{1}{2} \|b\xi\|^2 . \quad 1_{(0,\infty)}(b)$$

$$\int_{-\infty}^{0} \|e_\lambda \xi\|^2 |\lambda| \, d\lambda$$

$$= \int_{-\infty}^{0} d\lambda \, |\lambda| \int_{-\infty}^{\infty} \underset{1_{[\rho,\infty)}(\lambda)}{1_{(-\infty,\lambda]}(\rho)} \, d\|e_\rho \xi\|^2$$

$$= \int_{-\infty}^{0} d\|e_\rho \xi\|^2 \frac{\rho^2}{2} 1_{(-\infty,0]}(\rho)$$

$$\leq 32 \cdot 4 \int_{-\infty}^{\infty} \underbrace{\| e_\lambda \eta - \eta e_\lambda \|^2}_{= \| f_\lambda \eta - \eta f_\lambda \|^2} |\lambda| \, d\lambda$$

by Lem 3.16.

Thus $\exists \lambda \in \mathbb{R}$ s.t.

$$\| f_\lambda \xi \|^2 + \| f_\lambda \eta \|^2 + \delta_1^{-\frac{1}{2}} \| f_\lambda \xi - \xi f_\lambda \|^2$$

$$< 2^7 \| f_\lambda \eta - \eta f_\lambda \|^2$$

In particular $f_\lambda \neq 0$. putting $2^7 \delta_1^{-\frac{1}{2}} := \delta$ ∎

次の目標: $\xi \perp \eta$ の仮定をはずす.

後で射影族に関する Zorn の Lemma を使うとき,

$$p \xi p \quad と \quad p \eta p$$

を考えなくてはならないため.

Lem. 3.18

M III_1-factor w. Thm 3.9 (1)

$\xi \in P$ cyc & sep $\|\xi\| = 1$

$\eta \in H_{sa}$ $\|\eta\| = 1$ & $\eta \neq \pm\xi$

Let $\cancel{\ }0 < \theta < \pi$ be the angle:

$$\underset{\mathbb{R}}{\underset{\Uparrow}{\langle \xi, \eta \rangle}} = \underbrace{\|\xi\|}_{1} \underbrace{\|\eta\|}_{1} \cos\theta$$

Then $\forall \delta > 0$, $\exists p \in M^p \setminus \{0\}$

s.t.

- $\|p\xi\|^2 + \|p\eta\|^2 < \dfrac{2^{10}}{\sin^2\theta} \|p\eta - \eta p\|^2$
- $\|p\xi - \xi p\|^2 < \delta \|p\eta - \eta p\|^2$

proof

$$\eta = \cos\theta\, \xi + \sin\theta\, \overset{\circ}{\eta'}$$

$\to \|\eta'\| = 1$ $\xi \perp \eta'$.

$\eta' \in H_{sa}$.

Let $\delta_1 > 0$. take $p \neq 0$ proj in M.

s.t.

- $\|p\xi\|^2 + \|p\eta'\|^2 < 2^7 \|p\eta' - \eta' p\|^2$
- $\|p\xi - \xi p\|^2 < \delta_1 \|p\eta' - \eta' p\|^2$

Since $\sin\theta\, \eta' = \eta - \cos\theta\, \xi$,

$\sin\theta \|p\eta' - \eta' p\|$

$= \|p\eta - \eta p - \cos\theta (p\xi - \xi p)\|$

$\leq \|p\eta - \eta p\| + \underbrace{|\cos\theta|}_{\leq 1} \underbrace{\|p\xi - \xi p\|}_{\delta_1^{\frac{1}{2}}}$

$\rightsquigarrow \|p\eta - \eta p\| \geq (\sin\theta - \delta_1^{\frac{1}{2}}) \|p\eta' - \eta' p\|$.

Put $\delta_1 := \dfrac{\delta}{4} \sin^2\theta$

$\geq \dfrac{\sin\theta}{2} \|p\eta' - \eta' p\|$.

$\rightsquigarrow \quad \|p\xi - \xi p\|^2 < \delta_1 \times \dfrac{4}{\sin^2\theta} \|p\eta - \eta p\|^2$

$= \delta \|p\eta - \eta p\|^2$

<u>Note</u>

$\|p\eta\| \leq |\cos\theta| \|p\xi\| + |\sin\theta| \|p\eta'\|$

$\leq \sqrt{\|p\xi\|^2 + \|p\eta'\|^2}$,

Then

$\|p\xi\|^2 + \|p\eta\|^2 \leq 2\|p\xi\|^2 + 2\|p\eta'\|^2$

$< 2 \cdot 2^7 \|p\eta' - \eta' p\|^2$

$< \dfrac{2^{10}}{\sin^2\theta} \|p\eta - \eta p\|^2$

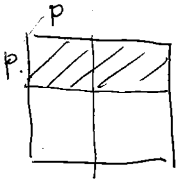
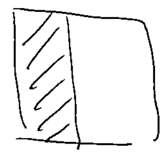
$\blacksquare$

$< \|p\xi\|^2 + \delta\|p\eta - \eta p\|^2$

$< (2^7 + \delta) \|p\eta - \eta p\|^2$

. <u>次の目標</u>　Zorn の Lemma で p を積み上げる.

$$\begin{cases} \|p\xi\|^2 + \|p\eta\|^2 < 2^7 \|p\eta - \eta p\|^2 \\ \|p\xi - \xi p\|^2 < \delta \|p\eta - \eta p\|^2 \end{cases}$$

$p\xi = $ 　　$\xi p = $ 

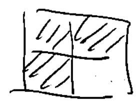
$\rightsquigarrow \quad p\xi - \xi p \overset{\xi-}{=} p\xi p - p^\perp \xi p^\perp$

$\|\underset{\sim}{\xi - p^\perp \xi p^\perp} - p\xi p\|^2 < \delta \|\underset{\sim}{\eta - p^\perp \eta p^\perp} - p\eta p\|^2$

余り. 小さくしたい.

$p^\perp M p^\perp \simeq M$ に注意して

$r \leq p^\perp$ を取り続けると…?

$\xi - p^\perp \xi p^\perp = $  $= p\xi + p^\perp \xi p$

$\|\xi - p^\perp \xi p^\perp\|^2 = \|p\xi\|^2 + \|p^\perp \xi p\|^2$

$\| \; p^\perp(\xi p - p\xi)$

Lem. 3.19

M : III_1-factor satisfying Thm 3.9 (1)

$\xi \in P$: cyc & sep. $\|\xi\| = 1$

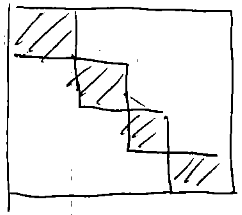
$\eta \in H_{sa}$: $\eta \perp \xi$ $\quad \|\eta\| = 1$

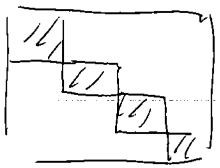
Then $\forall \delta > 0$ $\exists \{e_i\}_{i \in I}$ in M^p

s.t.

- $\sum_{i \in I} e_i = 1$
- $\|\xi - \sum e_i \xi e_i\|^2 \leq \delta$
- $\|\eta - \sum e_i \eta e_i\|^2 \geq 2^{-18}$

~~proof~~

$\rightsquigarrow$ $\xi \approx$  e_i たちと ほとんど commute

$\eta \not\approx$  e_i たちと ほとんど commute する ということはない.

proof

Let us introduce $\mathcal{F} \subseteq 2^{M^p}$ defined by

(1) $\emptyset \in \mathcal{F}$ $\quad (\rightsquigarrow \mathcal{F} \neq \emptyset)$

(2) $\{p_i\}_{i \in I} \in \mathcal{F}$

$\Leftrightarrow$ • $p_i \neq 0$ & $p_i \perp p_j$ if $i \neq j$

• For $p := 1 - \sum_{i \in I} p_i$,

$$\|\xi - p\xi p\|^2 + \|\eta - p\eta p\|^2 \leq 2^{14} \|\eta - p\eta p - \sum_i p_i \eta p_i\|^2$$

$$\|\xi - p\xi p - \sum_i p_i \xi p_i\|^2 \leq \delta \|\eta - p\eta p - \sum_i p_i \eta p_i\|^2$$

The order $=$ the inclusion

$\longrightarrow$ inductive

Let $\{q_i\}_{i\in I}$ be a maximal element of $\mathcal{F}$.

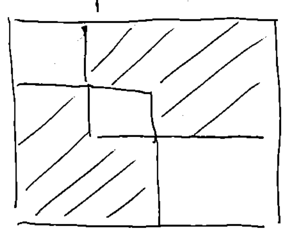
Put $q := 1 - \sum_{i\in I} q_i$

Want to show that $\{q_i\}_{i\in I} \cup \{q\}$ satisfies the statement of Lem.

Since $\{q_i\}_{i\in I}$,

$$\|\xi - q\xi q - \sum q_i \xi q_i\|^2$$

$$\leq \delta \|\eta - q\eta q - \sum q_i \eta q_i\|^2$$



$$\leq \delta \|\eta\|^2 = \delta.$$

Hence we must show.

$$\|\eta - q\eta q - \sum_i q_i \eta q_i\|^2 \geq 2^{-18}.$$

Suppose that

$$\|\eta - q\eta q - \sum_i q_i \eta q_i\|^2 < 2^{-18}.$$

We want to apply our argument to

$$(qMq,\ q\xi q,\ q\eta q)$$

not orthogonal.

Using

$$\|\xi - q\xi q\|^2 + \|\eta - q\eta q\|^2 < 2^{14}\cdot 2^{-18} = \frac{1}{16}$$

Thus $\|\xi - q\xi q\| < \frac{1}{4}$

$\|\eta - q\eta q\| < \frac{1}{4}$

$\rightsquigarrow q \neq 0$

☆ Angle of $\xi' := q\xi q$ & $\eta' := q\eta q^{-1}$

- $\|\xi'\| \geq \frac{3}{4}$, $\|\eta'\| \geq \frac{3}{4}$
- $\langle \xi', \eta' \rangle = \langle q\xi q, \eta \rangle$

$$= \underset{\overset{\|}{0}}{\langle \xi, \eta \rangle} - \langle (\xi - q\xi q), \eta \rangle$$

$\leadsto |\langle \xi', \eta' \rangle| \leq \|\xi - \xi'\| \, \|\eta\|^{\leq 1}$

$$< \frac{1}{4}$$

Let θ be the angle of ξ' & η'.

Then

$$|\cos\theta| = \frac{|\langle \xi', \eta' \rangle|}{\|\xi'\| \, \|\eta'\|} < \frac{1}{4} \times \frac{4}{3} \times \frac{4}{3} = \frac{4}{9}$$

$$< \frac{1}{2}$$

Thus $\sin^2\theta > \frac{3}{4}$

By Lem 1.8,

(qMq, qHq, qPq, qJq) std.

By Lem 1.11,

$q\xi q \in qPq$ is cyc & sep.

Since $q \sim 1$, $qMq \cong M$.

Hence we can apply ~~our argument~~ Lem to

$$\xi'' := \frac{\xi'}{\|\xi'\|} \in qPq, \quad \eta'' := \frac{\eta'}{\|\eta'\|} \in (qHq)_{sa}.$$

Thus $\exists r \in M^p \setminus \{0\}$ s.t.

$$\begin{cases} \bullet \ \|r\xi''\|^2 + \|r\eta''\|^2 < \dfrac{2^{10}}{\sin^2\theta} \|r\eta'' - \eta'' r\|^2 \\ \qquad < \dfrac{4}{3} \cdot 2^{10} \|r\eta'' - \eta'' r\|^2 \\ \bullet \ \|r\xi'' - \xi'' r\|^2 < \dfrac{\delta}{2} \|r\eta'' - \eta'' r\|^2 \end{cases}$$

Then

$$\begin{cases} \bullet \ \|r\xi'\|^2 + \|r\eta'\|^2 < \left(\dfrac{4}{3}\right)^3 \cdot 2^{10} \|r\eta' - \eta' r\|^2 \\ \qquad < 2^{13} \|r\eta' - \eta' r\|^2 \\ \bullet \ \|r\xi' - \xi' r\|^2 < \left(\dfrac{4}{3}\right)^2 \dfrac{\delta}{2} \|r\eta' - \eta' r\|^2 \\ \qquad < \delta \|r\eta' - \eta' r\|^2 \end{cases}$$

We will show that

$\{q_i\}_{i\in I} \cup \{r\} \in \mathcal{F}$,

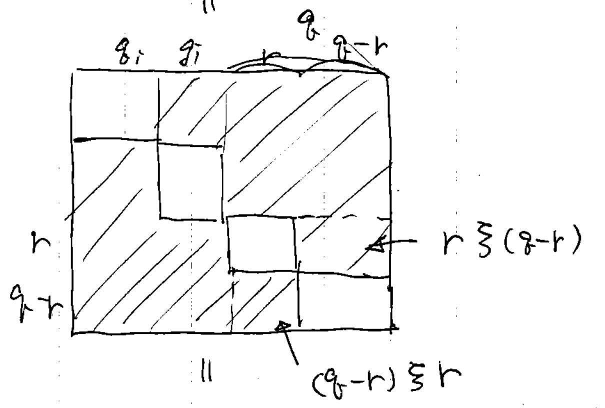
and contradiction.

What we have to prove is.

$$\begin{cases} \bullet \ \|\xi - (q-r)\xi(q-r)\|^2 + \|\eta - (q-r)\eta(q-r)\|^2 \\ \qquad \leqq 2^{14} \|\eta - (q-r)\eta(q-r) - r\eta r - \sum q_i \eta q_i\|^2 \quad \text{— (1)} \\ \bullet \ \|\xi - (q-r)\xi(q-r) - r\xi r - \sum q_i \xi q_i\|^2 \\ \qquad \leq \delta \|\eta - (q-r)\eta(q-r) - r\eta r - \sum q_i \eta q_i\|^2 \quad \text{— (2)} \end{cases}$$

$$\xi - (q-r)\xi(q-r) - r\xi r - \sum_i q_i \xi q_i$$

$$\|$$



$$\|$$

$$\xi - q\xi q - \sum_i q_i \xi q_i + \boxed{\begin{array}{l} r\xi(q-r) \\ +(q-r)\xi r \end{array}}$$

$$\cancel{r\, q\xi q}$$

$\rightsquigarrow$ LHS of ②

$$= \|\xi - q\xi q - \sum_i q_i \xi q_i\|^2$$

$$+ \|r\xi(q-r)\|^2 + \|(q-r)\xi r\|^2$$

$$\leqq \delta\,\|\eta - q\eta q - \sum_i q_i \eta q_i\|^2$$

$$+ \|r\,\underset{\xi'}{\underset{\|}{q\xi q}} - \underset{\xi''}{\underset{\|}{q\xi q}}\,r\|^2$$

$$< \delta\,\|\eta - q\eta q - \sum_i q_i \eta q_i\|^2$$

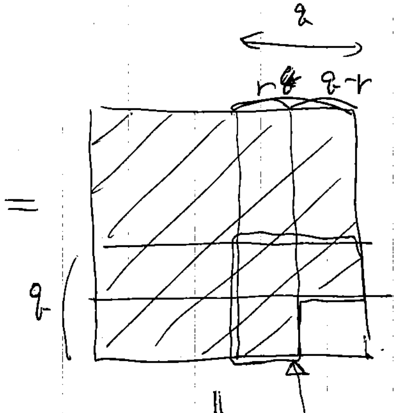
$$+ \delta\,\|r\,\underset{q\eta q}{\underset{\|}{\eta'}} - \underset{q\eta q}{\underset{\|}{\eta'}}\,r\|^2$$

$= \cancel{\delta}$ RHS of ②.

Thus we have proved ②.

①

$$\xi - (q-r)\xi(q-r) =$$



$$\|$$

$$\xi - q\xi q + q\xi r + r\xi(q-r)$$

$$\|\xi - (q-r)\xi(q-r)\|^2$$

$$= \|\xi - q\xi q\|^2 + \|q\xi r\|^2 + \|r\xi(q-r)\|^2$$

$$\|q\xi r\|^2 = \|r\xi q\|^2 = \|r\xi'\|^2$$

$$\|r\xi(q-r)\|^2 \leqq \|r\xi q\|^2 = \|r\xi'\|^2$$

$$\leq \| \xi - q \xi q \|^2 + 2 \| r \xi' \|^2$$

Similarly,

$$\| \eta - (q-r) \eta (q-r) \|^2$$

$$\leq \| \eta - q \eta q \|^2 + 2 \| r \eta' \|^2$$

$$\rightsquigarrow \| \xi - (q-r) \xi (q-r) \|^2$$

$$+ \| \eta - (q-r) \eta (q-r) \|^2$$

$$\leq \| \xi - q \xi q \|^2 + \| \eta - q \eta q \|^2$$

$$+ 2 \| r \xi' \|^2 + 2 \| r \eta' \|^2$$

$$\leq 2^{14} \| \eta - q \eta q - \sum_i q_i \eta q_i \|^2$$

$$+ 2^{14} \| r \eta' - \eta' r \|^2$$

$= \text{RHS of } (1).$ $\blacksquare$

Lem. 3.20

M : III_1 - factor satisfying Thm 3.9 (1).

$\xi \in \mathcal{P}$ cyc & sep $\|\xi\| = 1$

$\eta \in H$ $\eta \perp \xi$

Then $\forall \delta > 0$, $\exists p \in M^p \setminus \{0\}$ s.t.

- $\| p\xi - \xi p \|^2 \le \delta$
- $\| p\eta - \eta p \|^2 \ge 2^{-21} \|\eta\|^2$

proof

Suppose first $\eta \in H_{sa}$.

By Lem 3.19, $\exists \{e_i\}_{i \in I} \subset M^p$

s.t.

- $\sum_{i \in I} e_i = 1$
- $\| \xi - \sum_i e_i \xi e_i \|^2 \le \delta$
- $\| \eta - \sum_i e_i \eta e_i \|^2 \ge 2^{-18} \|\eta\|^2$

☆ M separable $\rightsquigarrow$ $|I|$ is countable.

Let $G := \prod_{i \in I} \{-1, 1\}$ cpct abel. group.

For $g \in G$,

$$U_g := \sum_{i \in I} g_i e_i \quad , \quad g = (g_i)_{i \in I}.$$

Then $U : G \longrightarrow B(H)$ is a unitary repn.

Let $\zeta \in H$. Then

$$\int_G u_g \zeta u_g^* \, dg = \sum_{ij} \int_G g_i \overline{g_j} \, e_i \zeta e_j \, dg$$

$$= \sum_{ij} \delta_{ij} \, e_i \zeta e_j$$

$$= \sum_i e_i \zeta e_i$$

Thus

$$\int_G (\eta - u_g \eta u_g^*) \, dg = \eta - \sum_i e_i \eta e_i$$

In particular,

$$\int_G \| \eta - u_g \eta u_g^* \| \, dg \geq \| \eta - \sum_i e_i \eta e_i \|$$

$$\geq 2^{-9} \| \eta \|$$

Thus for some $g \in G$,

$$\| \eta - u_g \eta u_g^* \| \geq 2^{-9} \| \eta \|$$

$$\|$$

$$\| u_g \eta - \eta u_g \|$$

Put $\xi' := \sum e_i \xi e_i$. Then $\| \xi - \xi' \|^2 < \delta$.

& $u_h \xi' = \xi' u_h \ \forall h \in G$.

Hence

$$\| u_g \xi - \xi u_g \| \leq \| u_g (\xi - \xi') \| + \underbrace{\| u_g \xi' - \xi' u_g \|}_{= 0} + \| (\xi' - \xi) u_g \|$$

$$\leq 2 \| \xi - \xi' \|$$

$$\leq 2 \delta^{1/2}.$$

Now $p := \frac{1}{2}(1+u_g)$ proj in M.

Then

- $\| p\xi - \xi p \|^2 \leq \delta$
- $\| p\eta - \eta p \|^2 \geq 2^{-20} \|\eta\|^2$

Done!

Let $\eta \in H$ in general.

$$\eta = \underset{\underset{\frac{\eta + J\eta}{2}}{\uparrow}}{\eta_1} + i \underset{\underset{\frac{\eta - J\eta}{2i}}{\uparrow}}{\eta_2}$$

$$\|\eta\|^2 = \|\eta_1\|^2 + \|\eta_2\|^2 - 2\mathrm{Re}\langle \eta_1, i\eta_2 \rangle$$

$$\langle \eta_1, i\eta_2 \rangle = -i \underbrace{\langle \eta_1, \eta_2 \rangle}_{\in \mathbb{R}}$$

$$= \|\eta_1\|^2 + \|\eta_2\|^2$$

Since $\langle J\eta, \xi \rangle = \langle J\xi, \eta \rangle$
$= \langle \xi, \eta \rangle$
$= 0$,

$\eta_1 \perp \xi$, $\eta_2 \perp \xi$.

Take $j \in \{1, 2\}$ so that

$$\|\eta_j\|^2 \geq \frac{1}{2} \|\eta\|^2.$$

By the first part of this proof,

$\exists p \in M^p \setminus \{0\}$, s.t.

$$\begin{cases} \cdot \; \| p\xi - \xi p \|^2 \leq \delta \\ \cdot \; \| p\eta_j - \eta_j p \|^2 \geq 2^{-20} \|\eta_j\|^2 \end{cases}$$

$$p\eta - \eta p = \underset{i H_{sa}}{\underset{\in}{(p\eta_1 - \eta_1 p)}} + \underset{H_{sa}}{\underset{\in}{\underbrace{i\,(p\eta_2 - \eta_2 p)}}}$$

$$\to \|p\eta - \eta p\|^2 = \|p\eta_1 - \eta_1 p\|^2 + \|p\eta_2 - \eta_2 p\|^2$$

$$\geq \|p\eta_j - \eta_j p\|^2$$

$$\geq 2^{-20}\|\eta_j\|^2$$

$$\geq 2^{-21}\|\eta\|^2$$

$\blacksquare$

Proof of Thm 3.9 (1) $\Rightarrow$ (2)

Let $a \in AB(\varphi)$

$$\mathring{a} := a - \varphi(a) 1_M \in M$$

$$\xi := \xi_\varphi \perp \mathring{a}\, \xi_\varphi =: \eta$$

$\exists\, p_n \in M^p \setminus \{0\}$ s.t.

$$\begin{cases} \cdot \ \| p_n \xi_\varphi - \xi_\varphi p_n \| \leq \dfrac{1}{n} \\ \cdot \ \| p_n \eta - \eta p_n \|^2 \geq 2^{-2} \| \eta \|^2 \end{cases}$$

$\rightsquigarrow (p_n) \in AC(\varphi)$ Prop 1.1

$$p_n \eta - \eta p_n$$

$$= p_n \mathring{a} \xi_\varphi - \mathring{a} \xi_\varphi p_n$$

$$= \underbrace{(p_n \mathring{a} - \mathring{a} p_n)}_{\xrightarrow{n\to\infty} 0} \xi_\varphi + \mathring{a} \underbrace{(p_n \xi_\varphi - \xi_\varphi p_n)}_{\xrightarrow{n\to\infty} 0}$$

Hence $\eta = 0$.

i.e. $a = \varphi(a) 1_M$

$\blacksquare$