Section 1 τ - measurable operators & measure topology.

§ 1.~~2~~ τ - measurable operators.

Setting

N, τ

semifinite vN alg.

faithful normal semifinite tracial weight.

i.e.

$\tau: N_+ \longrightarrow [0, \infty]$

tracial weight
$$\begin{cases} \cdot\ \tau(x+y) = \tau(x) + \tau(y) \\ \cdot\ \tau(\lambda x) = \lambda\tau(x) \\ \cdot\ \tau(x^*x) = \tau(xx^*) \end{cases}$$

· faithful $\tau(x) = 0 \Leftrightarrow x = 0$

· normal if $x_\alpha \nearrow x$ in N_+ then $\tau(x_\alpha) \nearrow \tau(x)$.

· semifinite

$$\{x \in M_+ \mid \tau(x) < \infty\} \subset M_+$$
σ-w dense.

We will fix

$$N \subset B(H)$$
&
$$\tau: N_+ \longrightarrow [0, \infty].$$

Affiliated Operators

A linear operator a on H means a map a with domain $D(a)$

$$a: D(a) \longrightarrow H.$$

a affiliated with M $(a \eta M)$

$\overset{\text{dfn}}{\Longleftrightarrow}$ $\forall y \in M'$ $ya \subseteq ay$

$\Longleftrightarrow$ $\forall u \in N'$ unitary $ua = au$

when $\underset{v|a|}{\overset{\|}{a}}$ is closed & densely defined

$$a \,\eta\, M \iff v \in M \ \&\ |a| = \int_0^\infty \lambda \, de(\lambda)$$

$$e(\lambda) \in N.$$

Defn 1.1.

$$\boxed{\begin{array}{l} a : \text{closed densely defn}, \ a \,\eta\, M \\ a \text{ is } \underline{\tau\text{-measurable}} \\ \overset{\text{defn}}{\iff} \displaystyle\lim_{\lambda\to\infty} \tau(1_{(\lambda,\infty)}(|a|)) = 0 \end{array}}$$

☆ $|a|$ has a "small tail".

☆ a τ-measurable

$$\iff \exists \lambda_0 > 0 \text{ s.t. } \tau(1_{(\lambda_0,\infty)}(|a|)) < \infty$$

since

$$1_{(\lambda_0,\infty)}(|a|) - 1_{(\lambda,\infty)}(|a|) \underset{\lambda\to\infty}{\nearrow} 1_{(\lambda_0,\infty)}(|a|).$$

☆ a $\overset{\text{closed, densely defn.}}{\tau\text{-meas.}} \iff a^*$ τ-meas

$$\iff |a| \ \tau\text{-meas}.$$

since

$a = v|a|$ polar dec.

$\to |a^*| = v|a|v^*.$

$$1_{(\lambda,\infty)}(|a^*|) = v\, 1_{(\lambda,\infty)}(|a|)\, v^*$$

$$\tau(1_{(\lambda,\infty)}(|a^*|)) = \tau(v^*v\, 1_{(\lambda,\infty)}(|a|))$$

$$= \tau(1_{(\lambda,\infty)}(|a|)).$$

No.

Notation.

$L^0(N,\tau) := \{ a \,\eta\, N \mid$ closed densely defn. & τ-measurable $\}$

✱ For lin. operators a & b on H, set

- $D(a+b) := D(a) \cap D(b)$

  $(a+b)\xi := a\xi + b\xi$, $\xi \in D(a+b)$

- $D(ab) := \{ \xi \in D(b) \mid b\xi \in D(a) \}$

  $ab\,\xi := a(b\xi)$, $\xi \in D(ab)$.

✱ For a closable operator a, $[a]$ denotes its closure. i.e. if $\xi \in D([a])$

$\exists\, \xi_n \in D(a)$ s.t.

$\xi_n \to \xi$ & $a\xi_n \to [a]\xi$.

Lem. 1.2

Let $a, b \in L^0(N,\tau)$

(1) $a+b$, ab (a^*) closable & densely defn affiliated with N

(2) $[a+b]$, $[ab] \in L^0(N,\tau)$

To prove this lemma, we rephrase the τ-measurability.

Lem 1.3.

Let $a \,\eta\, N$ closed & densely defn.

T.F.A.E.

(1) $a \in L^0(N,\tau)$

(2) $\forall \delta > 0 \ \exists\, p \in N^P$

s.t.

- $\|ap\| < \infty$ (i.e. $pH \subset D(a)$)
- $\tau(p^{\perp}) < \delta$

cf. Weyl characteristic values

$a \in K(H)_+$ eigenvalue

$\mu_1 \geqq \mu_2 \geqq \cdots$

<u>Proof of Lem 1.3.</u>

(1) $\Rightarrow$ (2) Put $p := 1_{\cancel{(\lambda,\infty)}\ [0,\lambda]}(|a|)$

for large λ.

(2) $\Rightarrow$ (1) Let $\delta > 0$

Take $p \in N^P$ s.t.

- $\|a p\| =: \lambda_0 < \infty$
- $\tau(p^\perp) < \delta$

Then

$$1_{(\lambda_0,\infty)}(|a|) \wedge p = 0$$

since $\xi \in 1_{(\lambda_0,\infty)}(|a|)H \cap pH$.

If $\xi \neq 0 \Rightarrow \||a|\xi\| > \lambda_0\|\xi\|.$

$$\begin{array}{c} \wedge\!\!| \\ \||a|p\|\,\|\xi\| \\ \| \\ \lambda_0\|\xi\| \end{array}$$

Thus $1_{(\lambda_0,\infty)}(|a|) \precsim p^\perp$ Point!

$\Big($ $e \wedge f = 0 \Rightarrow e \precsim f^\perp$

since $e = e - e\wedge f \underset{\text{in } N}{\sim} e \vee f - f \leqq 1 - f$ $\Big)$

Hence $\tau(1_{(\lambda_0,\infty)}(|a|)) < \delta.$

Proof of Lem 1.2.

$a+b$ densely defined?

Let $\delta > 0$. Take $p, q \in N^P$ s.t.

$$\|ap\| < \infty \ \& \ \tau(p^\perp) < \delta$$

$$\|bq\| < \infty \ \& \ \tau(q^\perp) < \delta.$$

Then $\underbrace{p \wedge q}_{= r} H \subset \underbrace{D(a) \cap D(b)}_{= D(a+b)}$

$\cancel{(a+b)p}$

$$\tau((p \wedge q)^\perp) = \tau(p^\perp \vee q^\perp)$$

$$\leq \tau(p^\perp) + \tau(q^\perp)$$

$$< 2\delta.$$

~~when~~ $\delta \downarrow 0$ then r

We have shown:

$\forall \delta > 0 \ \exists r \in N^P$ s.t. $rH \subset D(a+b)$, $\tau(r^\perp) < \delta$ ← τ-dense

This leads to $D(a+b) \subset H$ dense.

Write $E := D(a+b)$

For $n \geq 1$ take $r_n \in N^P$ s.t.

$$r_n H \subset E \ \& \ \tau(r_n^\perp) < \frac{1}{2^n}$$

$$e_n = r_n \wedge r_{n+1} \wedge \cdots \nearrow \text{increasing}$$

$$e_n^\perp = r_n^\perp \vee r_{n+1}^\perp \vee \cdots$$

$$\tau(e_n^\perp) \leq \sum_{k \geq n} \tau(r_k^\perp) < \frac{1}{2^{n-1}}$$

Hence $e_n \nearrow 1$ & $e_n H \subset r_n H \subset E$.

~~Pre~~ closable?

a, b densely defn $\rightsquigarrow$ a^*, b^*

τ-measurable $\rightsquigarrow$ a^*, b^* τ-meas.

Hence $a^* + b^*$ densely defined. operator.

It is trivial that $\Big\downarrow$ well-defn.

$$a + b \subseteq (a^* + b^*)^*$$

$\rightsquigarrow$ $a+b$ closable.

<u>ab</u> Take $p, q \in N^P$ as before.

Then

$$\{ \xi \in qH \mid bq\xi = pbq\xi \} = \ker(p^\perp b q) \cap qH$$

$p^\perp b q : H \to H$

$$x := p^\perp b q \in N$$
$$= w|x|.$$

$$\ker x = (\operatorname{ran} x^*)^\perp$$
$$= w^* w^\perp H$$

Hence

$$\{ \xi \in qH \mid bq\xi = pbq\xi \} = \underbrace{w^* w^\perp \wedge q}_{=: r} H$$

Then

- $abr = \underbrace{ap}_{\text{bdd}}\underbrace{bq}_{}r \rightsquigarrow \|abr\| < \infty$

  $rH \subset D(ab)$

- $$\tau(r^\perp) = \tau(w^* w \vee q^\perp)$$
  $$\leq \tau(w^* w) + \tau(q^\perp)$$
  $$= \tau(w w^*) + \tau(q^\perp)$$
  $$\leqq \tau(p^\perp) + \tau(q^\perp)$$
  $$< 2\delta$$

Thus $D(ab) \subset H$ dense.

ab is closable since

$$ab \subseteq \underbrace{(b^* a^*)}_{\substack{\uparrow \\ \text{well-defined!}}}{}^*$$

Affiliation

(2) $y \in N'$

$$y \cdot (a+b) = ya + yb \subseteq ay + by \subseteq (a+b)y$$

$$y \cdot ab \subseteq ayb \subseteq aby$$

(2) $[a+b]$, $[ab]$ η N.

This is easy:

$\left(\begin{array}{l} \text{if } a \,\eta\, N. \text{ closable, densely defn.} \\ \text{then } [a] \,\eta\, N \end{array} \right)$

Take $r \in N^p$ as (see the first part of proof)

- $rH \subset D(a+b) \subset D([a+b])$
- $\tau(r^\perp) < \delta$

By Lem 1.3. $[a+b] \in L^0(N, \tau)$

similarly $[ab] \in L^0(N, \tau)$

Hence $L^0(N, \tau)$ has binary operations:

$$(a, b) \longmapsto [a+b]$$
$$(a, b) \longmapsto [ab]$$
$$a \longmapsto a^*$$

Thm. 1.4.

$$\boxed{L^0(N, \tau) \text{ is a } *\text{-algebra}}$$

Lem 1.5

$a, b \in L^0(N, \tau)$

Suppose that $\exists E \underset{\text{subsp.}}{\subset} D(a) \cap D(b)$

s.t.

- E is τ-dense in H.

  (i.e. $\forall \delta > 0\ \exists p \in N^P$ s.t. $pH \subseteq E$ & $\tau(p^\perp) < \delta$)

- $a|_E = b|_E$

Then $a = b$

Proof of Thm 1.4.

$a, b, c \in L^0(N, \tau)$

$$[[a+b]+c] \overset{?}{=} [a + [b+c]]$$

$D([[a+b]+c])$ & $D([a+[b+c]])$ contains

$$D(a) \cap D(b) \cap D(c)$$

that is τ-dense.

Indeed, in the proof of Lem 1.2,

We have shown $D(a) \cap D(b) \subset H$ τ-dense.

thus so is $D(a) \cap D(b) \cap D(c) \subset H$.

(Since if $E_1, E_2 \subset H$ τ-dense

then $E_1 \cap E_2 \subset H$ τ-dense.

Let $\delta > 0$. Take $p, q \in N^P$

s.t. $pH \subset E_1$ $\tau(p^\perp) < \delta$

& $qH \subset E_2$ $\tau(q^\perp) < \delta$

then $p \wedge q H \subset E_1 \cap E_2$ & $\tau((p \wedge q)^\perp) < 2\delta$)

It is clear on $D(a) \cap D(b) \cap D(c)$

$[[a+b]+c]$ equals $[a+[b+c]]$.

Thus by Lem 1.5 we're done.

$\blacksquare$

Report 1. Show

$$[[ab]c] = [a[bc]]$$

No.

$$[a+b]^* = [a^* + b^*]$$

$$[ab]^* = [b^* a^*] \quad , a, b, c \in L^0(N, \tau)$$

Lem 1.6.

Let $p_1, p_2 \in N^p$

Suppose that

$$\left(\forall \delta > 0 \quad \exists p \in N \ \text{s.t.} \ \ p_1 \wedge p = p_2 \wedge p, \ \ \tau(p^\perp) < \delta \right)$$

Then $p_1 = p_2$

Proof.

$p_0 := p_1 - p_1 \wedge p_2$

$$p_0 \wedge p \leq \underset{\underset{p_2 \wedge p}{\parallel}}{p_1 \wedge p} \leq p_1 \wedge p_2$$

$\rightsquigarrow p_0 \wedge p = 0$

$\rightsquigarrow p_0 \precsim p^\perp \rightsquigarrow \tau(p_0) \leq \tau(p^\perp) < \delta$

Proof of Lem 1.5.

$M_2(N) \curvearrowright H \oplus H$

$t_2 := \mathrm{Tr} \otimes \tau$

graphs of a & b

Want to show $G(a) = G(b)$

i.e.

$p_a : H \oplus H \longrightarrow G(a)$

proj

$p_b : H \oplus H \longrightarrow G(b)$

then $p_a \overset{?}{=} p_b$.

Then $p_a, p_b \in M_2(N)$

Indeed $G(a)$ is $N' \otimes \mathbb{C}1$ -inv.

$$\left(\begin{array}{l} y \in N' \\ \begin{pmatrix} y & \\ & y \end{pmatrix} (\xi, a\xi) = (y\xi, ya\xi) \\ \qquad \qquad \qquad \qquad \underset{a \eta N}{=} (y\xi, ay\xi) \end{array} \right)$$

$\to \quad p_a \in (N' \otimes \mathbb{C}1)' = N \otimes M_2(\mathbb{C})$

Let $\delta > 0$.

Since $E \subset H$ is τ-dense.

$\exists\, q \in N^p$ s.t. $qH \subseteq E$

$\tau(q^\perp) < \delta$

Put $p := \begin{bmatrix} q & \\ & q \end{bmatrix} \in M_2(N)^p$

$$\tau_2(p^\perp) = \tau(q^\perp) + \tau(q^\perp) < 2\delta$$

$$\begin{aligned} & G(a) \cap (qH \oplus qH) \\ &= \{ (\xi, a\xi) \mid \xi \in \overset{E}{\overset{\cup}{qH}} \ \& \ a\xi \in qH \} \\ &= \{ (\xi, b\xi) \mid \xi \in qH \ \& \ b\xi \in qH \} \\ &= G(b) \cap (qH \oplus qH) \end{aligned}$$

$\leadsto \quad p_a \wedge p = p_b \wedge p$.

Applying Lem 1.6, we have $p_a = p_b$

<u>Notation</u>

We simply write $\begin{array}{c} a+b \\ ab \end{array}$ for $\begin{array}{c} [a+b] \\ [ab] \end{array}$ in $L^0(N, \tau)$.

No.

§1.2 Measure topology on $L^0(N, \tau)$

Let

$$V(\varepsilon, \delta) := \{ a \in L^0(N, \tau) \mid \tau(1_{(\varepsilon, \infty)}(|a|)) < \delta \}$$

$$= \{ a \in L^0(N, \tau) \mid \exists p \in N^P \text{ s.t. } \cdot \|ap\| < \varepsilon, \ \cdot \tau(p^\perp) < \delta \}$$

Lem 1.7.

(1) $V(\varepsilon, \delta)^* = V(\varepsilon, \delta)$

(2) $V(\varepsilon_1, \delta_1) + V(\varepsilon_2, \delta_2) \subset V(\varepsilon_1 + \varepsilon_2, \delta_1 + \delta_2)$

(3) $V(\varepsilon_1, \delta_1) V(\varepsilon_2, \delta_2) \subset V(\varepsilon_1 \varepsilon_2, \delta_1 + \delta_2)$

(4) $\varepsilon_1 < \varepsilon_2 \ \& \ \delta_1 < \delta_2 \Rightarrow V(\varepsilon_1, \delta_1) \subset V(\varepsilon_2, \delta_2)$

Proof.

(1). See the remark after Defn 1.1.

(2).(3) See the proof of Lem 1.2.

(4) Trivial.

Hence. $\{V(\varepsilon, \delta)\}_{\varepsilon, \delta > 0}$ defines a fundamental nbd of $0 \in L^0(N, \tau)$.

This topology is called the <u>measure topology</u>.

☆ $L^0(N, \tau)$ Hausdorff. 1st countable

i.e. $\bigcap_{\varepsilon, \delta > 0} V(\varepsilon, \delta) = \{0\}$

$$\left(\begin{array}{l} \ni a \\ \Rightarrow \tau(1_{(\varepsilon, \infty)}(|a|)) < \delta \\ \Rightarrow 1_{(\varepsilon, \infty)}(|a|) = 0. \ \forall \varepsilon \\ \quad |a| = 0 \end{array} \right)$$

☆ Lem 1.7 shows $L^0(N,\tau)$ is a topological $*$-algebra.

(↑ 1st countable.)

<u>Thm 1.8.</u>

(1) $N \subset L^0(N,\tau)$ dense.

(2) $L^0(N,\tau)$ is complete in meas. top.

<u>Proof</u>

(1) Let $a \in L^0(N,\tau)$

$\quad\quad = v|a|$.

$$|a| = \int_0^\infty \lambda \, de(\lambda).$$

$$a_n := \int_{[0,n]} \lambda \, de(\lambda).$$

Let $\varepsilon, \delta > 0$. Take $n_0 \in \mathbb{N}$ s.t. $\tau(1_{(n_0,\infty)}(|a|)) < \delta$.

Then if $n \geq n_0$. (& $n > \varepsilon$)

$$a_n - a = v \int_{(n,\infty)} \lambda \, de(\lambda)$$

$$\rightsquigarrow |a_n - a| = \int_{(n,\infty)} \lambda \, de(\lambda)$$

$$\rightsquigarrow 1_{(\varepsilon,\infty)}(|a_n - a|) = 1_{(n,\infty)}(|a_n - a|)$$

$$\leqq 1_{(n_0,\infty)}(|a_n - a|)$$

↓ trace $< \delta$.

Hence $n \geq n_0 \Rightarrow a_n - a \in V(\varepsilon, \delta)$. ┘

No.

(2) Let $a_n \in L^0(N, \tau)$ Cauchy seq.

By density $N \subset L^0(N, \tau)$

It suffices to show $a_n \longrightarrow {}^{\exists}a \in L^0(N, \tau)$

for $a_n \in N$.

WMA $a_{n+1} - a_n \in V(2^{-(n+1)}, 2^{-n})$

(Take a subseq).

Take $p_n \in N^P$ s.t.

- $\|(a_{n+1} - a_n) p_n\| < 2^{-(n+1)}$
- $\tau(p_n^\perp) < 2^{-n}$

(We may further assume $\|a_{n+1} p_n\|, \|a_n p_n\| < \infty$)

$q_n := p_{n+1} \wedge p_{n+2} \wedge \cdots \underset{n\to\infty}{\nearrow}$ increasing

$$\tau(q_n^\perp) \leqq \tau(p_{n+1}^\perp \vee p_{n+2}^\perp \vee \cdots) \leqq \sum_{k \geq n+1} \tau(p_k^\perp) \leqq 2^{-n} \rightsquigarrow q_n \nearrow 1.$$

For fixed $n \in \mathbb{N}$,

$\|(a_{m+1} - a_m) q_n\| \leqq 2^{-(m+1)}$ for $m \geqq n+1$.

$$\rightsquigarrow \|(a_{m+l} - a_m) q_n\| \leqq 2^{-(m+1)} + 2^{-(m+2)} + \cdots 2^{-(m+l)} \leqq 2^{-m}.$$

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Hence for $\xi \in \bigcup_{n\geq 1} q_n H$

$a_m \xi$ is Cauchy.

$$\begin{cases} a\xi := \lim_{m\to\infty} a_m \xi \in H \\ D(a) := \bigcup_{n\geq 1} q_n H. \quad \text{densely defn.} \end{cases}$$

Claim 1 a is closable.

Proof. Since a_m^* satisfies the same condition.

$\exists\, b\eta = \lim_{m\to\infty} a_m^* \eta$.

$\eta \in D(b) \underset{\text{dense}}{\subset} H$.

Then $\xi \in D(a)$ & $\eta \in D(b)$

$\langle a\xi, \eta\rangle = \lim_{m\to\infty} \langle a_m \xi, \eta\rangle$

$= \lim_{m\to\infty} \langle \xi, a_m^* \eta\rangle$ $\quad a_m$ bdd !

$= \langle \xi, b\eta\rangle$

$\Rightarrow \quad a \subseteq b^*$. ∎

Claim 2 $[a] \eta N$?

Proof. suffices to show $a \eta N$.

i.e. $\quad ya \subseteq ay \quad \forall y \in N'$

Let $\xi \in D(a) = \bigcup_n q_n H$

$\xi \in q_n H$

then $y\xi = y \underset{\substack{\uparrow \\ N}}{q_n} \xi = q_n y \xi \in q_n H \subset D(a)$

$ya\xi = \lim_{m\to\infty} y a_m \xi = \lim_{m\to\infty} a_m y \xi$

$= ay\xi$. ∎

Claim 3. $[a]$ τ-measurable?

i.e. $a_m - [a] \in V(\varepsilon, \delta)$ for $m \geq n_0$ ∎

Proof.

No.

- $q_n H \subset D(a) \subset D([a])$

$\Leftrightarrow$ $\| [a] q_n \| < \infty$

- $\tau(q_n^\perp) \leq 2^{-n}$

∎

Thus $[a] \in L^0(N, \tau)$.

Let $\varepsilon, \delta > 0$.

Take $n_0 \in \mathbb{N}$ s.t. $2^{-n_0} < \delta$, $2^{-n_0-1} < \varepsilon$.

Then If $m > n_0$ & $\xi \in H$ $\|\xi\| = 1$

$$\begin{aligned}
\| (a_m - [a]) q_{n_0} \xi \| &= \lim_{n \to \infty} \| (a_m - a_n) q_{n_0} \xi \| \\
&\leq 2^{-(m+1)} \cancel{\|\xi\|} \\
&\leq 2^{-(n_0+1)} \cancel{\|\xi\|} \\
&\leqq \varepsilon \cancel{\|\xi\|}
\end{aligned}$$

$\tau(q_{n_0}^\perp) \leq 2^{-n_0} < \delta$.

Section 2 $L^p(M)$

§2.1 Setting.

M : vN alg (possibly type III) $\curvearrowright H$

φ_0 : f.n.s. weight.

$\varphi_0 : M_+ \to [0,\infty]$

- $\varphi_0(x+y) = \varphi_0(x) + \varphi_0(y)$
- $\varphi_0(\lambda x) = \lambda \varphi_0(x)$
- If $x_\alpha \nearrow x$ then $\varphi_0(x_\alpha) \nearrow \varphi_0(x)$
- $\{x \in M_+ \mid \varphi_0(x) < \infty\} \subset M_+$ σ-w dense.

$\rightsquigarrow$ $\sigma^{\varphi_0} : \mathbb{R} \to \mathrm{Aut}(M)$ modular auto. grp.

$\rightsquigarrow$ $M \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \curvearrowright H \otimes L^2(\mathbb{R})$

$$\| $$

$\pi(M) \vee \{\lambda^{\varphi_0}(t)\}''_{t\in\mathbb{R}}$

$$\begin{cases} \cdot\ (\pi(x)\xi)(s) := \sigma^{\varphi_0}_{-s}(x)\,\xi(s) \\ \cdot\ (\lambda^{\varphi_0}(t)\xi)(s) := \xi(-t+s) \end{cases}$$

Dual flow θ

$\exists\, \theta : \mathbb{R} \to \mathrm{Aut}(M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})$ action.

$$\begin{cases} \theta_s(\pi(x)) = \pi(x) \\ \theta_s(\lambda^{\varphi_0}(t)) = e^{-ist}\lambda^{\varphi_0}(t) \end{cases}$$

$\rightsquigarrow$ $(M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})^\theta = \pi(M)$.

Operator valued wt T_θ

$$T_\theta : (M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})_+ \longrightarrow \widehat{\pi(M_+)}$$

extended positive part.

$=$ "{positive self adj. op affiliated with M}"

$\int_0^\infty \lambda \, de(\lambda) + \infty p^\perp$

$e(\lambda) \nearrow p \quad \lambda \nearrow \infty$

$$T_\theta(x) := \int_{\mathbb{R}} \theta_s(x) \, ds \qquad x \in (M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})_+$$

$$\theta_t(T_\theta(x)) = T_\theta(x)$$

$$\rightsquigarrow T_\theta(x) \in \pi\widehat{(M_+)}$$

Dual weights

$$W_{ns}(M) \longrightarrow W_{ns}(M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})$$

$$\varphi \longmapsto \hat{\varphi}$$

$$\hat{\varphi} := \varphi \circ \pi^{-1} \circ T_\theta$$

$\rightsquigarrow$ • $S(\hat{\varphi}) = \pi(S(\varphi))$

• $\hat{\varphi} \cdot \theta_t = \hat{\varphi}$.

Conversely if $\psi \in W_{ns}(M \rtimes_{\sigma^{\varphi_0}} \mathbb{R})$

$\psi \cdot \theta_s = \psi$

then $\exists! \varphi \in W_{ns}(M)$ s.t. $\psi = \hat{\varphi}$

• If $\varphi \in W_{fns}(M)$ then

• $\sigma_t^{\hat{\varphi}}(\pi(x)) = \pi(\sigma_t^{\varphi}(x))$

• $\sigma_t^{\hat{\varphi}}(\lambda^{\varphi_0}(s)) = \pi([D\varphi : D\varphi_0 \circ \sigma_s^{\varphi_0}]_t) \lambda^{\varphi_0}(s)$

No.

Canonical trace τ

$M \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$ $\quad \hat{\varphi}_0$ dual wt.

$$\rightsquigarrow \begin{cases} \sigma_t^{\hat{\varphi}_0}(\pi(x)) = \pi(\sigma_t^{\varphi_0}(x)) \\ \qquad = \lambda^{\varphi_0}(t)\, \pi(x)\, \lambda^{\varphi_0}(t)^* . \\ \sigma_t^{\hat{\varphi}_0}(\lambda^{\varphi_0}(s)) = \lambda^{\varphi_0}(s) \end{cases}$$

$$\to \sigma_t^{\hat{\varphi}_0}(x) = \lambda^{\varphi_0}(t)\, x\, \lambda^{\varphi_0}(t)^*$$
$$\forall x \in M \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$$

Let h_{φ_0} pos. self-adj. η $M \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$

s.t. $\lambda^{\varphi_0}(t) = h_{\varphi_0}^{it}$ $\quad \forall t \in \mathbb{R}$

Set $\tau := \hat{\varphi}_0 h_{\varphi_0}^{-1}$

$\tau(x) =$ " $\hat{\varphi}_0(h_{\varphi_0}^{-1} x)$ "

Then τ trace on $M \rtimes_{\sigma^{\varphi_0}} \mathbb{R}$

$$\theta_s(\lambda^{\varphi_0}(t)) = e^{-ist} \lambda^{\varphi_0}(t)$$

$$\to \theta_s(h_{\varphi_0}) = e^{-s} h_{\varphi_0}$$

$$\tau \cdot \theta_s = \hat{\varphi}_0 h_{\varphi_0}^{-1} \cdot \theta_s$$
$$= (\hat{\varphi}_0 \circ \theta_s) \cdot \theta_s(h_{\varphi_0}^{-1})$$
$$= e^{-s} \tau$$

$\tau \cdot \theta_s = e^{-s} \tau$ $\quad$ trace scaling.

☆ The triple $(M \rtimes_{\sigma^{\varphi_0}} \mathbb{R}, \theta, \tau)$ is indep from φ_0. Namely, let $\varphi_1 \in W_{fns}(M)$

$$M \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \xrightarrow{\exists \Psi} M \rtimes_{\sigma^{\varphi_1}} \mathbb{R}$$

s.t. $\Psi(\pi(x)) = \pi(x)$ $\quad x \in M$

$\Psi(\lambda^{\varphi_0}(t)) = \pi([D\varphi_0 : D\varphi_1]_t)\, \lambda^{\varphi_1}(t)$

$\Psi \cdot \theta_s = \theta_s \cdot \Psi$

$\tau_1 \circ \Psi = \tau_0$

Notation.

$$N := M \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \curvearrowright H \otimes L^2(\mathbb{R})$$

$$\tau := \widehat{\varphi_0} \cdot h_{\varphi_0}^{-1}$$

$$h_{\varphi_0}^{it} = \lambda^{\varphi_0}(t)$$

$$L^0(N, \tau) := \{ a \,\eta\, N \mid \text{closed densely defn. } \& \ \tau\text{-measurable} \}$$

Trivially,

$$\begin{array}{c} N \subset L^0(N,\tau) \\ \cup \\ M \end{array}$$

(we omit $\pi_{\sigma^{\varphi_0}}$ the embedding $M \hookrightarrow N$)

☆ $\exists\ M_* \hookrightarrow L^0(N,\tau)$ linear embedding

Let us start from M_*^+.

Take $\varphi \in M_*^+ \rightsquigarrow \hat{\varphi} \in W_{ns}(N)$

$\rightsquigarrow$ $\exists!\ h_\varphi \,\eta\, N$ positive self-adj.

s.t. $\hat{\varphi} = \tau h_\varphi$

h_φ is τ-measurable ?

Lem 2.1.

$\boxed{\forall \varphi \in W_{ns}(M) \quad \forall \lambda > 0 \qquad \tau(1_{(\lambda,\infty)}(h_\varphi)) = \frac{1}{\lambda}\varphi(1)}$

☆ Hence for $\varphi \in W_{ns}(M)$,

h_φ is τ-meas iff $\varphi(1) < \infty$
i.e.
$\varphi \in M_*^+$

Proof.

well-defn on $S(h_\varphi)=S(\varphi)$

$$\tau(1_{(\lambda,\infty)}(h_\varphi)) = \hat{\varphi}(h_\varphi^{-1} 1_{(\lambda,\infty)}(h_\varphi))$$

$$= \varphi(T_\theta(h_\varphi^{-1} 1_{(\lambda,\infty)}(h_\varphi))) \quad -(*)$$

$$T_\theta(h_\varphi^{-1} 1_{(\lambda,\infty)}(h_\varphi))$$

$$= \int_{\mathbb{R}} ds\ \theta_s(h_\varphi^{-1} 1_{(\lambda,\infty)}(h_\varphi))$$

$$= \int_{\mathbb{R}} ds\ e^s h_\varphi^{-1} \underbrace{1_{(\lambda,\infty)}(e^s h_\varphi)}_{=\,1_{(e^{-s}\lambda,\infty)}(h_\varphi)}$$

$$= \int_{\mathbb{R}} ds\ e^s \int_{[0,\infty)} de(t)\ \underbrace{t^{-1} 1_{(e^s\lambda,\infty)}(t)}_{=\,t^{-1}1_{(-\infty,\log\frac{t}{\lambda})}(s)}$$

$e^s\lambda < t$

$s < \log\frac{t}{\lambda}$

$[0,\infty) \leftarrow S(h_\varphi)$

$$= \int_{[0,\infty)} de(t)\ t^{-1} \int_{\mathbb{R}} ds\ e^s 1_{(-\infty,\log\frac{t}{\lambda})}(s)$$

$$= \int_{[0,\infty)} de(t)\ \frac{1}{\lambda} = \frac{1}{\lambda} S(h_\varphi) = \frac{1}{\lambda} S(\varphi)$$

Hence

$$(*) = \frac{1}{\lambda}\varphi(1)$$

So, we have a map.

$$\begin{array}{ccc} M_*^+ & \longrightarrow & L^0(N,\tau)_+ \\ \psi & & \psi \\ \varphi & \longmapsto & h_\varphi \end{array}$$

☆ This map ~~is bijective~~ satisfies

- $h_{\varphi+\psi} = h_\varphi + h_\psi$
- $h_{\lambda\varphi} = \lambda h_\varphi$

Indeed,

$$\widehat{\varphi+\psi} = \hat{\varphi} + \hat{\psi}$$

$$\tau(h_{\varphi+\psi}\,\cdot\,) \qquad \tau(h_\varphi\,\cdot\,) + \tau(h_\psi\,\cdot\,)$$

$$= \tau(h_\varphi \dotplus h_\psi\,\cdot\,)$$

(This is the form sum, which coincides with $[h_\varphi + h_\psi]$ since $D(h_\varphi^{1/2}) \cap D(h_\psi^{1/2}) \subset H \otimes L^2(\mathbb{R})$ dense)

$$\rightsquigarrow \quad h_{\varphi+\psi} = h_\varphi + h_\psi$$

Thus the map $\varphi \mapsto h_\varphi$ extends to

$$M_* \longrightarrow L^0(N,\tau)$$
$$\varphi \longmapsto h_\varphi$$

This map ~~satis~~ satisfies the M-M bimodularity

$$h_{x\varphi} = x h_\varphi$$

$$h_{\varphi y} = h_\varphi y \qquad x, y \in N$$

$$\left(\begin{aligned} \widehat{x\varphi x^*} &= x\varphi x^* \cdot T_\theta \\ &= x(\varphi \cdot T_\theta)x^* \qquad \theta_s(x) = x \\ &= x\hat{\varphi}x^* \\ \rightsquigarrow \quad h_{x\varphi x^*} &= x h_\varphi x^* \end{aligned} \right)$$

No.

☆ $|h_\varphi| = h_{|\varphi|}$

$\varphi = v|\varphi|$ p.d. $v^*v = S(|\varphi|)$

$\to h_\varphi = h_{v|\varphi|} = v\, h_{|\varphi|}$

positive self-adj.

$v^*v = S(|\varphi|) = S(h_{|\varphi|})$

Hence $h_\varphi = v\, h_{|\varphi|}$ is p.d.

In particular.

$$\begin{array}{ccc} M_* & \longrightarrow & L^0(N,\tau) \\ \psi & & \psi \\ \varphi & \longmapsto & h_\varphi \end{array}$$

is injective.

Recall $\theta_s(h_\varphi) = e^{-s} h_\varphi$

Lem. 2.2

$\boxed{M_* \xrightarrow{h} \{a \in L^0(N,\tau) \mid \theta_s(a) = e^{-s}a\} \text{ is bijective}}$

Proof

If $\theta_s(a) = e^{-s}a$, $a = v|a|$

satisfies

$\theta_s(v) = v \leadsto v \in M$

$\theta_s(|a|) = e^{-s}|a|$

$\tau|a|$ θ-inv

$\| $

$\hat{\varphi}$ $\exists \varphi \in W_{ns}(M)$

Hence $|a| = h_\varphi$ is τ-meas. & $\varphi(1) < \infty$

$a = v h_\varphi = h_{v\varphi}$

Also, we have

$$M = \{ a \in L^0(a.\tau) \mid \theta_s(a) = a \ \forall s \}$$

If $\theta_s(a) = a$, $a = v|a|$

$v \in M$. $|a| = \int_0^\infty \lambda \, de(\lambda)$, $e(\lambda) \in M$

$$\tau(1_{(\lambda,\infty)}(|a|)) \ll 1 \text{ if } \lambda \gg 1$$

$$\| $$

$$\tau(\theta_s(1_{(\lambda,\infty)}(|a|)))$$

$$\| $$

$$e^{-s} \tau(1_{(\lambda,\infty)}(|a|)) \quad \forall s$$

Hence $\tau(1_{(\lambda,\infty)}(|a|)) = 0$

i.e. $|a|$ is bdd.

$\boxed{\begin{array}{l} \textbf{Defn. 2.3} \\ 0 < p \leq \infty \\ L^p(M) := \{ a \in L^0(N.\tau) \mid \theta_s(a) = e^{-\frac{s}{p}} a \} \end{array}}$

$$L^\infty(M) = M$$

$$L^1(M) \overset{\text{linear}}{\cong} M_*$$

$$h_\varphi \in L^1(M) \longleftarrow \varphi \in M_*$$

No.

§2.2 Tracial functional tr & the L^p-norm.

Set

$$\begin{array}{ccc} \mathrm{tr}: L^1(M) & \longrightarrow & \mathbb{C} \\ \Psi & & \Psi \\ h_\varphi & \longmapsto & \varphi(1) \end{array}$$

&

$$\|a\|_1 = \mathrm{tr}(|a|) \quad a \in L^1(M).$$

Then

$$\|h_\varphi\|_1 = \mathrm{tr}(|h_\varphi|) = \mathrm{tr}(h_{|\varphi|}) = |\varphi|(1) = \|\varphi\|.$$

$\rightsquigarrow$ $\|\cdot\|_1$ is a norm &

$$M_* \xrightarrow{h} L^1(M)$$

is surjective & isometric.

NOTE: $|\mathrm{tr}(a)| \leqq \|a\|_1$

Now, let $a \in L^p(M)$ $0 < p < \infty$.

$$\theta_s(a) = e^{-\frac{s}{p}} a$$

If $a = v|a|$ p.d. then

$$v \in M \ \& \ \theta_s(|a|) = e^{-\frac{s}{p}}|a|$$

$$\Downarrow$$

$$\theta_s(|a|^p) = e^{-s}|a|^p$$

i.e. $|a|^p \in L^1(M)$.

(NOTE. $a \in L^0(N,\tau)_+ \Rightarrow a^p \in L^0(N,\tau)_+$)

$$\|a\|_p := \mathrm{tr}(|a|^p)^{1/p} \quad a \in L^p(M).$$

Thm. 2.4 $(L^p(M), \|\cdot\|_p)$ is a Banach sp.

Lem. 2.5
$1 \leq p < \infty$, $\varepsilon, \delta > 0$.

$$V(\varepsilon, \delta) \cap L^p(M) = \{ a \in L^p(M) \mid \|a\|_p \leq \varepsilon \delta^{1/p} \}$$

Proof.

$$a \in L^p(M) \rightsquigarrow |a|^p \in L^1(M)$$

Thus $\exists \varphi \in M_*^+$ $|a|^p = h_\varphi$

$$\begin{aligned}
\tau(1_{(\varepsilon,\infty)}(|a|)) &= \tau(1_{(\varepsilon^p,\infty)}(|a|^p)) \\
&= \tau(1_{(\varepsilon^p,\infty)}(h_\varphi)) \\
&= \frac{1}{\varepsilon^p} \varphi(1) \\
&= \frac{1}{\varepsilon^p} \mathrm{tr}(h_\varphi) \\
&= \varepsilon^{-p} \|a\|_p^p
\end{aligned}$$

∎

Lem. 2.6
On $L^1(M)$, the norm top = the measure top

Proof.
Use Lem 2.5. ∎

Set

$$\mathbb{C}_+ := \{ \alpha \in \mathbb{C} \mid \mathrm{Re}\,\alpha \geq 0 \}$$

$$\mathbb{C}_+^\circ := \{ \alpha \in \mathbb{C} \mid \mathrm{Re}\,\alpha > 0 \}$$

Lem. 2.7
Let $h \in L^0(N, \tau)_+$.
Then the map $\mathbb{C}_+^\circ \ni \alpha \longmapsto h^\alpha \in L^0(N, \tau)$
is ~~differentiable~~

✼ Note $|h^\alpha| = h^{\overset{>0}{\mathrm{Re}\,\alpha}} \in L^0(N, \tau)$
well-defn.

(1) $\|h\| < \infty \rightsquigarrow h \in N_+$

$$\mathbb{C}_+^0 \ni \alpha \longmapsto h^\alpha \in N$$

is differentiable w.r.t. the norm top.

$$\frac{d}{d\alpha} h^\alpha = h^\alpha \log h \in N$$

(2) $h \in L^0(N,\tau)_+$

Note $h^\alpha \log h \in L^0(N,\tau)$.

$$\left(|h^\alpha \log h| = \int_{[0,\infty)} \underset{\substack{\uparrow \\ \text{increasing.}}}{\lambda^{\mathrm{Re}\,\alpha} \log \lambda}\, de(\lambda) \right)$$

Let $\varepsilon, \delta > 0$.

Take $\lambda > 0$ s.t. $\tau(1_{(\lambda,\infty)}(h)) < \delta$

$$\left(\begin{array}{c} \| \\ p^\perp \in N^P \end{array} \right)$$

$$p := 1_{[0,\lambda]}(h) \in N$$

Since $\|hp\| < \infty$,

$$\left\| \left(\frac{1}{\alpha - \alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0} \log h \right) p \right\| < \varepsilon$$

if $\alpha \approx \alpha_0$

i.e. $\frac{1}{\alpha - \alpha_0}(h^\alpha - h^{\alpha_0}) - h^{\alpha_0} \log h \in V(\varepsilon, \delta)$

Set

$$S := \{ \alpha \in \mathbb{C} \mid 0 \leqq \mathrm{Re}\,\alpha \leqq 1 \}$$

$$S^0 := \{ \alpha \in \mathbb{C} \mid 0 < \mathrm{Re}\,\alpha < 1 \}$$

Lem. 2.8

Let $h, k \in L^1(M)_+$

(1) $h^\alpha k^{1-\alpha} \in L^1(M) \quad \forall \alpha \in S^0$

(2) $S^0 \ni \alpha \longmapsto h^\alpha k^{1-\alpha} \in L^1(M)$ is differentiable (w.r.t. $\|\cdot\|_1$)

Proof.

(1) $\underset{\substack{\cap \\ L^0(N,\tau)}}{\theta_s(h^\alpha k^{1-\alpha})} = e^{-\alpha s} e^{-(1-\alpha)s} h^\alpha k^{1-\alpha}$

$= e^{-s} h^\alpha k^{1-\alpha}$

$\rightsquigarrow h^\alpha k^{1-\alpha} \in L^1(M)$

(2) $S^0 \ni \alpha \longmapsto h^\alpha \in L^0(N,\tau)$

$S^0 \ni \alpha \longmapsto k^{1-\alpha} \in L^0(N,\tau)$

are differentiable. (Lem 2.7)

$\rightsquigarrow S^0 \ni \alpha \longmapsto h^\alpha k^{1-\alpha} \in L^0(N,\tau)$

differentiable w.r.t. meas. top

$\|$

$\|\ \|_1$-top

on $L^1(M)$

∎

Lem. 2.9.

$\boxed{\begin{array}{l} \text{Let } t \in \mathbb{R}. \text{ Suppose } a, b \in L^0(N,\tau) \\ \text{satisfies} \\ \quad \theta_s(a) = e^{-(\frac{1}{2}+it)s} a \\ \quad \theta_s(b) = e^{-(\frac{1}{2}+it)s} b \quad \forall s \in \mathbb{R}. \\ \text{Then } b^*a,\ ab^* \in L^1(M) \ \& \\ \quad tr(b^*a) = tr(ab^*) \end{array}}$

Proof.

By polarization, we may assume $a = b$.

Then $a^*a \in L^1(M)_+$ and $\exists \varphi \in M_*^+$ s.t.

$a^*a = h_\varphi$.

$tr(a^*a) = \varphi(1) \underset{\text{Lem 2.1}}{=} \tau(1_{(1,\infty)}(h_\varphi))$

$= \tau(1_{(1,\infty)}(a^*a))$

$= \tau(1_{(1,\infty)}(aa^*))$

$= tr(aa^*)$

∎

No.

Lem. 2.10

$1 \leq p, q \leq \infty$, $1/p + 1/q = 1$

Let $a \in L^p(M)$, $b \in L^q(M)$.

(1) ab, $ba \in L^1(M)$

(2) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$

Proof.

(1) Easy.

(2) If $p = 1$. take $\varphi \in M_*$, $a = h_\varphi$.
$b \in M$

then

$$\mathrm{tr}(ab) = \mathrm{tr}(h_{\varphi_b}) = \varphi(b)$$
$$= \mathrm{tr}(ba)$$

Suppose that $1 < p, q < \infty$.

WMA $a \in L^p(M)_+$, $b \in L^q(M)_+$

Put $h := a^p$, $k := b^q \in L^1(M)_+$

and

$$F(\alpha) := \mathrm{tr}(h^\alpha k^{1-\alpha}) \qquad \alpha \in S^0$$
$$G(\alpha) := \mathrm{tr}(k^{1-\alpha} h^\alpha)$$

$\underset{\substack{\text{Lem 2.8} \\ \& \\ |\mathrm{tr}(a)| \leq \|a\|_1}}{\rightsquigarrow}$ F, G hol on S^0

For $\alpha = 1/2 + it$, $t \in \mathbb{R}$,

$$\theta_s(h^{1/2+it}) = e^{-(1/2+it)s} h^{1/2+it}$$
$$\theta_s(k^{1/2+it}) = e^{-(\frac{1}{2}+it)s} k^{1/2-it}.$$

$$\underset{\text{Lem 2.9}}{\rightsquigarrow} F(1/2+it) = \mathrm{tr}(h^{1/2+it} k^{1/2-it})$$
$$= \mathrm{tr}(h^{1/2+it} (k^{1/2+it})^*)$$
$$= \mathrm{tr}((k^{(1/2+it)})^* h^{1/2+it})$$
$$= G(1/2+it)$$

$\rightsquigarrow F = G$, $\mathrm{tr}(ab) = F(1/p) = G(1/p) = \mathrm{tr}(ba)$ ∎

Lem 2.11

$$\boxed{\begin{array}{l}\text{Let } h, k \in L^1(M)_+ ,\ \|h\|_1 = 1 = \|k\|_1 \\ \text{Then} \quad \|h^\alpha k^{1-\alpha}\|_1 \leqq 1 \quad \forall \alpha \in S^0\end{array}}$$

Proof.

Put $s := \mathrm{Re}\,\alpha \qquad 0 < s < 1.$

$t := \mathrm{Im}\,\alpha$

$\rightsquigarrow \quad h^s \in L^{1/s}(M) \;\&\; \|h^s\|_{1/s} = 1$

$\underset{\text{Lem 2.5}}{\rightsquigarrow} \quad h^s \in V(s^{-s}, s)$

$k^{1-s} \in V((1-s)^{-(1-s)}, 1-s)$

$\underset{\text{Lem 1.7}}{\rightsquigarrow} \quad h^s k^{1-s} \in V(s^{-s}(1-s)^{-(1-s)}, 1)$

$\cap$

$L^1(M)$

$\underset{\text{Lem 2.5}}{\rightsquigarrow} \quad \|h^\alpha k^{1-\alpha}\|_1 = \|h^s k^{1-s}\|_1$

$\leqq s^{-s}(1-s)^{-(1-s)}$

$\star \quad (0,1) \ni s \longmapsto s^{-s}(1-s)^{-(1-s)}$ bdd.

$\rightsquigarrow$ The 3 lines theorem for $h^\alpha k^{1-\alpha}$ on $\{\alpha \in \mathbb{C} \mid \varepsilon \leqq \mathrm{Re}\,\alpha \leqq 1-\varepsilon\}$ is applied.

$\rightsquigarrow \quad \|h^\alpha k^{1-\alpha}\|_1 \leqq \varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)}$

$\forall \alpha,\ \varepsilon \leqq \mathrm{Re}\,\alpha \leqq 1-\varepsilon$

Since $\varepsilon^{-\varepsilon}(1-\varepsilon)^{-(1-\varepsilon)} \underset{\substack{\varepsilon \to 0 \\ \varepsilon \to 1}}{\longrightarrow} 1$

$\rightsquigarrow \quad \|h^\alpha k^{1-\alpha}\|_1 \leqq 1. \quad \forall \alpha \in S^0$

∎

Thm. 2.12 (Hölder inequality)

Let $p, q \in [1, \infty]$, $1/p + 1/q = 1$

Then

$$\|ab\|_1 \leqq \|a\|_p \|b\|_q$$

Proof.

When $p = 1$. easy.

Let $1 < p, q < \infty$

Suppose $\|a\|_p = 1 = \|b\|_q$.

$a = u|a|$

$b = |b^*| v$ p.d. $u, v \in M$

$|a|^p \in L^1(M)_+ \ni |b^*|^q$

$$\|ab\|_1 = \| u|a| |b^*| v \|_1$$

$$\leqq \| |a| |b^*| \|_1$$

$$= \| (|a|^p)^{1/p} (|b^*|^q)^{1/q} \|_1$$

$$\leqq 1 \quad \text{Lem 2.11.}$$

$\blacksquare$

Lem 2.13

$1 \leq p, q \leq \infty$, $1/p + 1/q = 1$

Then

$$\|a\|_p = \sup\{ |tr(ab)| \mid b \in L^q(M),\ \|b\|_q \leq 1 \}$$

$\forall a \in L^p(M)$

Proof

When $p = 1$, easy.

Suppose $1 < p, q < \infty$. & $\|a\|_p = 1$.

$$|tr(ab)| \leqq \|ab\|_1 \leqq \|a\|_p \|b\|_q \leqq 1.$$

$\forall b \in L^q(M)$, $\|b\|_q \leq 1$.

Next put

$b := |a|^{p/q} u^*$, $a = u|a|$ p.d.

$\|b\|_q = \| |a|^{p/q} \|_q = 1$.

& $tr(ab) = tr(u|a| |a|^{p/q} u^*) = tr(u|a|^p u^*)$

$= tr(|a|^p) = 1.$

$\blacksquare$

Proof of Thm 2.4.

- Lem 2.13 shows $\|\cdot\|_p$ is a norm.
- Lem 2.5 $\rightsquigarrow$ $\|\cdot\|_p$ − top = meas. top on $L^p(M)$.

Now let $a_n \in L^p(M)$ $\|\ \|_p$− Cauchy seq.

- Then this is a meas. Cauchy & $\exists\, a \in L^0(N,\tau)$.

$a_n \longrightarrow a$ in meas. top as $n \to \infty$.

It is now clear that

$$\theta_s(a) = e^{-\frac{s}{p}} a . \qquad \blacksquare$$

☆ $t \in \mathbb{R}$

$$\mathcal{H}_t := \{ a \in L^0(N,\tau) \mid \theta_s(a) = e^{-(\frac{1}{2}+it)s} a \}$$

$$\langle a, b \rangle := tr(b^* a) = tr(ab^*) \quad \text{Lem 2.9.}$$

$$\mathcal{H}_t = \lambda^{\varphi_0}(t)\, L^2(M)$$

$\rightsquigarrow$ Hilbert sp.

Report

$$\boxed{\begin{array}{l} \text{Show } a \in L^p(M),\ b \in L^q(M). \quad 1/p + 1/q = 1/r,\ r \geq 1 \\ \|ab\|_r \leq \|a\|_p \|b\|_q \end{array}}$$

☆ This ineq. is valid for $\forall\, p, q, r > 0$.
See Kosaki.

☆ Hint: Prove Lem 2.11 in the $L^r(M)_+$ setting.

§2.3 Reflexivity of $L^p(M)$

Thm. 2.14 (Clarkson inequality)

$2 \leqq p < \infty$

Then

$$\|a+b\|_p^p + \|a-b\|_p^p \leqq 2^{p-1}(\|a\|_p^p + \|b\|_p^p) \quad \forall a, b \in L^p(M)$$

☆ $L^p(M)$ $(2 \leqq p < \infty)$ is unif. convex.

(i.e. $\forall\, 0 < \varepsilon < 1$ $\exists \delta > 0$ s.t.

$\forall x, y \in L^p(M)$, $\|x\| = 1 = \|y\|$ & $\|x-y\| \geq \varepsilon$

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta \text{)}$$

☆ Hence $L^p(M)$ $2 \leqq p < \infty$ reflexive

(Milman–Pettis, Kakutani, Ringrose)

easy proof.

Say X unif. convex Banach sp.

We will show $\iota : X \to X^{**}$ surjective.

Let $y \in X^{**}$, $\|y\| = 1$.

Take $x_\alpha \in X$ $\|x_\alpha\| \leqq 1$ s.t.

$x_\alpha \longrightarrow y$ in $\sigma(X^{**}, X^*)$

Then

$$\left\|\frac{x_\alpha + x_\beta}{2}\right\|_X \xrightarrow[\alpha,\beta]{} 1.$$

~~Loc~~ Unif convexity

$\rightsquigarrow$ $\|x_\alpha - x_\beta\| \xrightarrow[\alpha,\beta]{} 0$

Hence x_α is norm-convergent in X,

and $y \in X$. ∎

Thm. 2.15

$1 \leq p < \infty$, $\quad 1/p + 1/q = 1$

(1) $\forall\, a \in L^q(M)$,

$$\varphi_a(b) := \mathrm{tr}(ab) \quad b \in L^p(M)$$

is a norm bdd lin. functional.

(2)
$$\begin{array}{cccc} \varphi: & L^q(M) & \longrightarrow & L^p(M)^* \\ & \Psi & & \Psi \\ & a & \longmapsto & \varphi_a \end{array}$$

is a surjective isometric lin. map.

<u>Proof.</u>

(1) See Lem 2.13.

(2) From Lem 2.13, $\|\varphi_a\|_{L^p(M)^*} = \|a\|_q$

$$\underset{\substack{\uparrow \\ \text{Banach.}}}{\varphi : L^q(M)} \hookrightarrow L^p(M)^* \quad \text{isometric.}$$

The image of φ is w^*-dense. also by Lem 2.13.

If $p \geq 2$, we know $L^p(M)$ reflexive.

$$\begin{array}{cc} L^p(M)^{**} & \\ \| & L^p(M)^* \\ L^p(M) & \end{array}$$

Hence the w^*-closure of $\varphi(L^q) = L^p(M)^*$

$$\|$$

the norm closure of $\varphi(L^q)$

$$\|$$

$$\varphi(L^q).$$

If $p < 2$, we have $q > 2$. and

$$L^p(M) \xrightarrow{\sim} L^q(M)^*$$

$$\rightsquigarrow L^p(M)^* \xrightarrow{\sim} L^q(M)^{**} \cong L^q(M).$$

$\blacksquare$

Proof of Thm 2.14

Set $P := M \oplus M \curvearrowright H \oplus H$

$\psi_0 := \varphi_0 \oplus \varphi_0$

$Q := P \rtimes_{\sigma^{\psi_0}} \mathbb{R}$

$= M \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \oplus M \rtimes_{\sigma^{\varphi_0}} \mathbb{R} \curvearrowright (H \otimes L^2(\mathbb{R}))^{\oplus 2}$

$\rightsquigarrow$

$\tau_Q = \tau_N \oplus \tau_N$

$\theta^Q_s = \theta^N_s \oplus \theta^N_s$

$P_* = M_* \overset{\ell^1}{\oplus} M_*$

$h_{(\varphi,\psi)} = h_\varphi \oplus h_\psi$

$L_0(\widetilde{Q}, \tau^Q) = L^0(N, \tau_N) \oplus L^0(N, \tau_N)$

$tr = tr \oplus tr$

L^p ~~$(Q, \tau$~~

$L^p(P) = L^p(M) \oplus L^p(M)$

$\|(a,b)\|_p^p = \|a\|_p^p + \|b\|_p^p.$

Want to show:

$\|(a+b, a-b)\|_p^p \leqq 2^{p-1} \|(a,b)\|_p^p$

i.e.

$\|(a+b, a-b)\|_p \leqq 2^{1/q} \|(a,b)\|_p.$

We may assume $\|(a,b)\|_p = 1.$

From Lem 2.13, suffices to show

$|tr((a+b, a-b)\cdot(c,d))| \leqq 2^{1/q}$ — (*).

with $(c,d) \in L^q(M \oplus M)$

$\|(c,d)\|_q = 1.$

Now let

$a = u h^{1/p}$, $b = v k^{1/p}$ left p.d.

$c = f^{1/q} w$ $d = g^{1/q} z$ right p.d.

$\to h, k, f, g \in L^1(M)_+$

$\|(h,k)\|_1 = 1 = \|(f,g)\|_1$

Put

$$F(\alpha) := \mathrm{tr}\Big((uh^\alpha + vk^\alpha) f^{1-\alpha} w + (uh^\alpha - vk^\alpha) g^{1-\alpha} z \Big)$$

$$\alpha \in S^\circ$$
$$(\text{i.e. } 0 < \mathrm{Re}\,\alpha < 1)$$

▷ $F(\alpha)$ is hol. (Lem 2.8)

▷ $F(1/p) = \mathrm{tr}\big((a+b)c + (\underset{a-b}{\cancel{f.g}}). d \big)$

(= the left had side $\cancel{\text{of}}$ $(*)$)

▷ $\cancel{|F(\alpha)| \leq}$ $|F(\alpha)| \leq 2$ (Lem 2.11)

Claim. $|F(1/2 + it)| \leq \sqrt{2} \quad \forall t \in \mathbb{R}$.

$$F(\alpha) = \mathrm{tr}\big((u,-v) \cdot (h^\alpha, k^\alpha)(f^{1-\alpha}, g^{1-\alpha})(w,z) \big)$$
$$+$$
$$\mathrm{tr}\big((v,u) \cdot (k^\alpha, h^\alpha)(f^{1-\alpha}, g^{1-\alpha})(w,z) \big)$$

Proof of Claim. $\in \mathcal{H}_t^{\oplus 2}$

$$|F(1/2+it)|^2 = \Big| \mathrm{tr}\Big((uh^{1/2+it} + vk^{1/2+it}, uh^{1/2+it} - vk^{1/2+it}) \cdot (f^{1/2-it} w, g^{1/2-it} z) \Big) \Big|^2$$

かける

$$\leq \| (uh^{1/2+it} + vk^{1/2+it}, uh^{1/2+it} - vk^{1/2+it}) \|_2^2 \cdot \| (f^{1/2-it} w, g^{1/2-it} z) \|_2^2$$

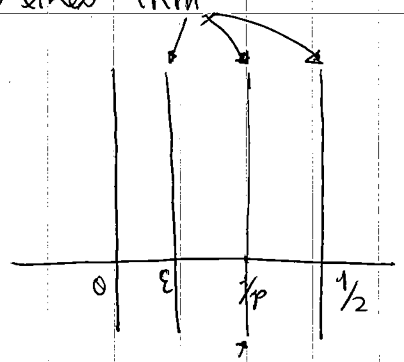
$$= \big(2\|uh^{1/2+it}\|_2^2 + 2\|vk^{1/2+it}\|_2^2 \big) \cdot \big(\|f^{1/2-it} w\|_2^2 + \|g^{1/2-it} z\|_2^2 \big)$$

$$= \underbrace{\big(2\|h^{1/2}\|_2^2 + 2\|k^{1/2}\|_2^2 \big)}_{=2} \cdot \underbrace{\big(\|f^{\frac{1}{2}}\|_2^2 + \|g^{1/2}\|_2^2 \big)}_{=1}$$

$$= 2$$

∎

Apply the 3-lines thm



$$1/p = t\varepsilon + (1-t)\tfrac{1}{2}$$

with $t = \dfrac{1/2 - 1/p}{1/2 - \varepsilon}$

$$|F(1/p)| \leq \sup_{s\in\mathbb{R}} |F(\varepsilon + is)|^{t} \cdot \sup_{s\in\mathbb{R}} |F(1/2 + is)|^{1-t}$$

$$\leq 2^{t} \sqrt{2}^{\,1-t}$$

$$= 2^{\frac{1/2 - 1/p}{\frac{1}{2}-\varepsilon}} \sqrt{2}^{\,\frac{1/p - \varepsilon}{\frac{1}{2}-\varepsilon}}$$

$$\xrightarrow{\varepsilon \to 0} 2^{1-\frac{2}{p}} \; 2^{1/p} = 2^{1/8}$$

$\blacksquare$

Section 3 Applications

§3.1 Standard forms

$1 \leqq p \leqq \infty$.

$L^p(M)$ is an M-M-bimodule:

$$\lambda_p(x)a := xa$$
$$\rho_p(x)a := ax \qquad a \in L^p(M),\ x \in M.$$

☆ $\|\lambda_p(x)\| \leqq \|x\|$

$\|\rho_p(x)\| \leqq \|x\|$.

$a \in L^p(M) \cap V(\|a\|_p, 1)$. Lem 2.5

& $xV(\varepsilon, \delta) \subset V(\|x\|\varepsilon, \delta)$

$\rightsquigarrow xa \in L^p(M) \cap V(\|x\|\|a\|_p, 1)$

i.e. $\|xa\|_p \leqq \|x\|\|a\|_p$.

☆ Conjugate unitary

$$\begin{array}{cccc} J_p : & L^p(M) & \longrightarrow & L^p(M) \\ & \cup & & \cup \\ & a & \longmapsto & a^* \end{array}$$

$\rightsquigarrow \|a^*\|_p = \|a\|_p$.

Lem. 3.1

(1) $\lambda_p, \rho_p : M \to B(L^p(M))$ faithful repn. (ρ_p : anti repn)

(2) $J_p \lambda_p(x) J_p = \rho_p(x^*) \quad {}^\forall x \in L^\infty(M)$.

(3) ${}^\forall z \in Z(M)$, $\lambda_p(z) = \rho_p(z)$.

Proof.

(1) If $\lambda_p(x) = 0$, then

${}^\forall a \in L^p(M)$, $b \in L^q(M)$

$\mathrm{tr}(xab) = \mathrm{tr}((\lambda_p(x)a) \cdot b) = 0$

$\rightsquigarrow x = 0$ since $L^1(M) = L^2(M) \cdot L^2(M)$.

(2) $J_p \lambda_p(x) J_p a = J_p \lambda_p(x) a^*$

$= J_p x a^*$

$= a x^*$

$= \rho_p(x^*) a$

(3) (Trivial. If $p = \infty$.

If $p = 1$.)

Trivial. since $Z(M) \subset Z(N)$ $\blacksquare$

<u>Lem.</u> 3.2

$$\boxed{\begin{array}{l} 1 \leqq p \leqq \infty. \\ \lambda_p(M)' = \rho_p(M) \ \& \ \rho_p(M)' = \lambda_p(M) \end{array}}$$

<u>Proof.</u>

It is clear that

$$\lambda_p(M) \subseteq \rho_p(M)'$$

$$\rho_p(M) \subseteq \lambda_p(M)'$$

We will show

$$\lambda_p(M) \supseteq \rho_p(M)' \overset{J_p \cdot J_p}{\Longleftrightarrow} \rho_p(M) \supseteq \lambda_p(M)'.$$

(i) $p = \infty$. Let $T \in \lambda_\infty(M)'$. put $a := T(1)$

Then $\forall\, b \in M$

$$T(b) = T(b \cdot 1)$$

$$= b\, T(1)$$

$$= \rho_\infty(a) b.$$

(ii) $p = 1$. Let $S \in \lambda_1(M)'$

$$S : L^1(M) \to L^1(M)$$

$$T := {}^tS : L^\infty(M) \to L^\infty(M)$$

Then $T \in \rho_\infty(M)'$

$$T\rho_\infty(x)\, a \overset{?}{=} \rho_\infty(x)\, T a \qquad \forall x \in M,\ \forall a \in L^\infty(M)$$

$\forall\, b \in L^1(M)$

$$\begin{aligned}
tr(T\rho_\infty(x)a \cdot b) &= tr(\rho_\infty(x) a\, S(b)) \\
&= tr(a x\, S(b)) \\
&= tr(a\, \lambda_1(x) S(b)) \\
&= tr(a\, S\, \lambda_1(x) b) \\
&= tr(T(a)\, \underset{\underset{x b}{\shortparallel}}{\lambda_1(x)\, b}) \\
&= \cancel{tr}\, tr(\rho_\infty(x)(T(a))\, b).
\end{aligned}$$

Thus $T = \lambda_\infty(y)$ $\quad {}^\exists y \in M$.

Then $S = \rho_1(y)$

$$\begin{aligned}
tr(\underset{\underset{M}{\Uparrow}}{a}\, \rho_1(y)\, \underset{\underset{L^1(M)}{\Uparrow}}{b}) &= tr(\underset{\underset{L^1}{\Uparrow}}{ab}\, y) = tr(y\, ab) \\
&= tr(\lambda_\infty(y) a\, b) = tr(T(a)\, b) \\
&= tr(a\, S(b))
\end{aligned}$$

(iii) $1 < p < \infty$.

Let $T \in \lambda_p(M)'$.

$$T : L^p(M) \longrightarrow L^p(M).$$

Want to define

$$S : L^1(M) \longrightarrow L^1(M)$$

by

$$S\Big(\sum_{i=1}^n b_i a_i\Big) = \sum_{i=1}^n b_i\, T(a_i)$$

$a_i \in L^p(M)$, $\quad b_j \in L^q(M)$.

$(1/p + 1/q = 1)$

Well-defn?

Suppose $\displaystyle\sum_{i=1}^n \cancel{a_i b_i}\ b_i a_i = 0 \overset{?}{\Rightarrow} \sum_{i=1}^n b_i\, T(a_i) = 0.$

Put

$$a := \left(\sum_{i=1}^n a_i^* a_i\right)^{1/2} \in L^p(M)_+$$

$x_i := a_i a^{-1}$ on supp(a).

$$x_i^* x_i = a^{-1} a_i^* a_i a^{-1}$$

$$\sum_{i=1}^n x_i^* x_i = a^{-1} a^2 a^{-1} = s(a)$$

$\rightarrow$ x_i bdd & $x_i \in M$

Then

$$\left(\sum_{i=1}^n b_i x_i\right) a = \sum_{i=1}^n b_i a_i = 0$$

$\rightarrow$ $a_i s(a) = a_i$

$$\sum_{i=1}^n b_i x_i = 0.$$

$$\rightsquigarrow \sum_{i=1}^n b_i T(a_i) = \sum_{i=1}^n b_i T(x_i a)$$

$$= \sum_{i=1}^n b_i x_i T(a) = 0.$$

$\rightsquigarrow$ $S : L^1(M) \to L^1(M)$ well-defn. lin. map

Bounded?

$\forall c \in L^1(M)$ $\exists a \in L^p(M)$. $\exists b \in L^q(M)$

s.t. $c = \cancel{ab}\ ba$ & $\|c\|_1 = \|a\|_p \|b\|_q$

$$\|S(c)\|_1 = \|b T(a)\|_1$$

$$\leq \|b\|_q \|T(a)\|_p \quad \text{Hölder}$$

$$\leq \|T\|_* \|b\|_q \|a\|_p$$

$$= \|T\| \|c\|_1.$$

OK

Finally $S \in \lambda_1(M)'$

$$S(\lambda_1(x)c) = S(xc) = S(xba)$$

$$= xbT(a)$$

$$= \lambda_1(x) S(ba)$$

$$= \lambda_1(x) S(c)$$

Hence $\exists y \in M$ $S = \rho_1(y)$,

$$bT(a) = S(c) = cy = bay. \quad \forall b \in L^q,\ a \in L^p$$

∎

No.

Lem 3.3 (Positive Cone)

$$\boxed{\begin{gathered} 1 \leqq p, q \leqq \infty, \quad 1/p + 1/q = 1. \\ L^p(M)_+ = \{ a \in L^p(M) \mid tr(ab) \geqq 0 \quad \forall\, b \in L^q(M)_+ \} \end{gathered}}$$

Proof.

When $p, q \in \{1, \infty\}$. well-known.

Let $1 < p, q < \infty$

It is clear that "$\subseteq$".

since if $a \in L^p(M)_+$

$$tr(ab) = tr(\underset{\in L^1(M)}{a b^{1/2}}\, \underset{\in L^{2q}(M)}{b^{1/2}})$$

$$= tr(\underbrace{b^{1/2}\, a\, b^{1/2}}_{\in L^1(M)_+}) \quad \text{Lem 2.10}.$$

Suppose $a \in L^p(M)$ satisfies

$$tr(ab) \geqq 0 \quad \forall\, b \in L^q(M)_+$$

$\rightarrow a^* = a$.

(Since for $b \in L^q(M)_+$

$$tr(a^* b) = \overline{tr(ba)} = \overline{tr(ab)} = tr(ab) \geqq 0)$$

$$a = a_+ - a_-$$

$$a_+ := \underset{\in L^p(M)_+}{\frac{|a| + a}{2}}, \quad a_- := \underset{\in L^p(M)_+}{\frac{|a| - a}{2}}$$

$$\underset{L^0(N,\tau)}{a_+ a_-} = 0.$$

Let $b := a_-^{p/q} \rightarrow b \in L^q(M)_+$

$$0 \leqq tr(ab) = tr((a_+ - a_-) b)$$

$$= - tr(a_-^p)$$

$$\leqq 0$$

$\rightarrow tr(a_-^p) = 0$

$\rightarrow a_-^p = 0$ i.e. $a_- = 0$

Thm. 3.4. (Standard form)

$$\boxed{(\lambda_2(M), L^2(M), J_2, L^2(M)_+) \text{ is a standard form of } M.}$$

Proof.

We know.

- $\lambda_2 : M \to B(L^2(M))$ norm bdd.
- $J_2 \lambda_2(M) J_2 = \rho_2(M) = \lambda_2(M)'$ (Lem 3.2)
- $J_2 \xi = \xi$ for $\xi \in L^2(M)_+$ (Trivial)
- $L^2(M)_+$ is self-dual. (Lem 3.3)

λ_2 is normal?

$$\left(\operatorname{tr}(\lambda_p(x) \underset{L^p(M)}{\underset{\cap}{a}} \overset{L^q(M)\ \downarrow}{b}) = \operatorname{tr}(\underset{L^1(M)}{\underset{\cap}{xab}}) = \langle x, ab \rangle \qquad 1/p + 1/q = 1 \right)$$

$\lambda_2(x^*) = \lambda_2(x)^*$.

$$\langle \lambda(x^*) a, b \rangle = \operatorname{tr}(b^* x^* a) = \operatorname{tr}((\lambda(x) b)^* a)$$
$$= \langle a, \lambda(x) b \rangle \qquad \blacksquare$$

★ $L^2(M)$ is very concrete!

But note that.

$$L^2(M) \subset L^0(N, \tau)$$

★ $\forall \varphi \in M_*^+ \ \exists! \xi \in L^2(M)_+ \quad \varphi(x) = \langle \lambda_2(x) \xi, \xi \rangle$

Indeed. $\xi := h_\varphi^{1/2}$ is OK.

& ξ^2 must equal h_φ.

§3.2 Modular theory

φ : f.n. state on M.

$\rightsquigarrow h_\varphi \in L^1(M)$. $\quad \theta_s(h_\varphi) = e^{-s} h_\varphi$

(Recall $\hat{\varphi} = \tau h_\varphi$)

Then we have

$$\sigma_t^\varphi(x) = h_\varphi^{it} x h_\varphi^{-it} \quad x \in M.$$

(This follows from $\sigma_t^{\hat{\varphi}} = \sigma_t^\varphi$ on M)

We can check φ & σ^φ satisfy the KMS condition.

Let

$$F(\alpha) := \mathrm{tr}(h_\varphi^{1+i\alpha} x h_\varphi^{-i\alpha} y)$$

$0 \leq \mathrm{Im}\,\alpha \leq 1$.

well-defn, since

$\mathrm{Re}(1+i\alpha) > 0$, $\longrightarrow h_\varphi^{1+i\alpha}, h_\varphi^{-i\alpha} \in L^0(N,\tau)$.

$\mathrm{Re}(-i\alpha) > 0$.

& $h_\varphi^{1+i\alpha} x h_\varphi^{-i\alpha} y \in L^1(M)$.

$F(\alpha)$ analytic on $0 < \mathrm{Im}\,\alpha < 1$

(cf Lem 2.7, 2.8)

$$F(t) = \mathrm{tr}(h_\varphi^{1+it} x h_\varphi^{-it} y)$$

$$= \varphi(\alpha_t(x) y) \quad \leftarrow \text{banded.}$$

$$F(t+i) = \mathrm{tr}(\underbrace{h_\varphi^{it} x h_\varphi^{-it}}_{M} \underbrace{{}^{+1} y}_{L^1(M)})$$

$$= \mathrm{tr}(h_\varphi y \alpha_t(x))$$

$$= \varphi(y \alpha_t(x)).$$

Report.

φ, ψ : f.n. states. Show

$$[D\psi : D\varphi]_t = h_\psi^{it} h_\varphi^{-it}.$$

(or

$$\sigma_t^{\psi,\varphi}(x) = h_\psi^{it}\, x\, h_\varphi^{-it}.$$
)

Some properties of std. form

☆ $L^2(M)_+ \longrightarrow M_*^+$ is a bijection

$\xi \longmapsto \langle \cdot\, \xi, \xi \rangle$

inj

$$\langle \cdot\, \xi, \xi\rangle = tr(\cdot\, \xi^2)$$

$$\| \text{ if } \rightsquigarrow \xi^2 = \eta^2$$

$$\langle \cdot\, \eta, \eta\rangle = tr(\cdot\, \eta^2) \qquad \text{in } L^1(M)_+$$

$$\rightsquigarrow \xi = \eta.$$

surj Let $\varphi \in M_*^+ \rightsquigarrow h_\varphi \in L^1(M)_+$

$$\to h_\varphi^{1/2} \in L^2(M)_+$$

$$\varphi(x) = \langle x h_\varphi^{1/2}, h_\varphi^{1/2} \rangle.$$

$\lrcorner$

☆ Araki – Powers – Størmer ineq:

$$\|\xi-\eta\|^2 \leqq \|\omega_\xi - \omega_\eta\|_{M_*} \leqq \|\xi-\eta\|\,\|\xi+\eta\|$$

easy

$$\|\omega_\xi - \omega_\eta\|_{M_*} = \| h_{\omega_\xi} - h_{\omega_\eta} \|_{L^1(M)}$$

$$\omega_\xi - \omega_\eta = \frac{1}{2}\omega_{\xi+\eta,\xi-\eta} + \frac{1}{2}\omega_{\xi-\eta,\xi+\eta}$$

$$h_{\omega_\xi} = \xi^2.$$

$$\|\xi-\eta\|_2^2 \leqq \|\xi^2-\eta^2\|_1.$$

$$e_+ := 1_{[0,\infty)}(\xi-\eta)$$

$$e_- := 1_{(-\infty,0)}(\xi-\eta) \qquad e_+ + e_- = 1.$$

$$\|\xi-\eta\|^2 = tr((\xi-\eta)(\xi-\eta))$$

$$= tr((\xi-\eta)(\xi-\eta)e_+) \leqq tr((\xi+\eta)\boxed{(\xi-\eta)e_+}) = tr((\xi-\eta)(\xi+\eta)e_+)$$

$$+$$

$$tr((\eta-\xi)(\eta-\xi)e_-) \leqq tr((\eta+\xi)(\eta-\xi)e_-)$$

$$= \frac{1}{2}\, tr(((\xi+\eta)(\xi-\eta)+(\xi-\eta)(\xi+\eta))e_+)$$

$$+$$

$$\frac{1}{2}\, tr(((\eta+\xi)(\eta-\xi)+(\eta-\xi)(\eta+\xi))e_-)$$

$$= \mathrm{tr}\left((\xi^2-\eta^2)e_+ + (\eta^2-\xi^2)e_-\right) = \mathrm{tr}\big(\underbrace{(\xi^2-\eta^2)}_{\in L^1(M)}(e_+-e_-)\big)$$

$$\leq \mathrm{tr}(|\xi^2-\eta^2|)$$

☆ $\mathrm{Aut}(M, J, P, \underset{L^2(M)}{H}) = \{\mathrm{id}_H\}$.

$U : H \to H$ unitary

- $JU = UJ$
- $Ux = xU \quad \forall x \in M$
- $U L^2(M)_+ = L^2(M)_+$

$U \in M' = JUJ \rightsquigarrow U \in Z(M)$

$JUJ = U^*$ self-adj unitary

$U = e_+ - e_-$

$\xi \in L^2(M)_+ \rightsquigarrow U\xi \geq 0$

$$e_- U\xi e_- \geq 0$$
$$\|$$
$$-e_-\xi e_- \leq 0$$

$\to e_-\xi e_- = 0$

$\to e_-\xi^{1/2} = 0.$

$\to e_-\xi = 0$

Since $L^2(M) = \mathrm{span}\, L^2(M)_+$

$e_-\xi = 0 \quad \forall \xi \in L^2(M)$

$e_- = 0.$

No.

Kosak:

Lem. Hölder

Lem 3.5

$$x \in L^0(N,\tau)_+$$

$$a \in N \quad \|a\| \leq 1$$

Then $a^* x^p a \leq (a^* x a)^p$

$$\forall p \leq 1$$

Pf

if $\|x\| < \infty$ ok.

(Hansen)

if $x \in L^0(N,\tau)_+$

$$x_n = \sum_0^n x \, 1_{[0,n]}(x)$$

$x_n \to x$ in meas

$x_n^p \to x^p$ in meas.

$$a^* x_n^p a \leq (a^* x_n a)^p \leq (a^* x a)^p$$

$L^0(N,\tau)_+ \ni x \longmapsto x^p$ cont ??

$$x^p - y^p$$

$\|xe\| < \infty$. $\|ye\| < \infty$. $\tau(e^\perp) < \delta$.

$\|(x-y)e\| < \varepsilon$.

Lem 3.6

$x, y \in L^0(M)_+ \quad 0 < p \leq 1.$

$$\mathrm{tr}((x+y)^p) \leq \mathrm{tr}(x^p + y^p)$$

Pf $x \leq x+y$

y

$x^{1/2} = u(x+y)^{1/2}$ $\quad x = u(x+y)u^*$

$y^{1/2} = v(x+y)^{1/2}$ $\quad y = v(x+y)v^*$

$$\mathrm{tr}(x^p + y^p) \geq \mathrm{Tr}(u(x+y)^p u^* + v(x+y)^p v^*)$$

$u^*u + v^*v = s(x+y)$

Lem 3.7 $x \in L^p(M)$. $0 < p \leq \infty$.

$\exists\, w \in M^{PI}$ s.t.

$$\mathrm{Re}(x)_+ \leq w|x|w^*.$$

Pf.

$x = u|x|$. $u \in M$

$e = \mathrm{supp}\,\mathrm{Re}(x)_+ \in M$.

$a := e \cdot \frac{1+u}{2}$

$a|x|^{1/2} = w\left|a|x|^{1/2}\right|$.

$$a|x|a^* = w\left|a|x|^{1/2}\right|^2 w^*$$

$$= w|x|^{1/2} a^* a |x|^{1/2} w^*$$

$$\leq w\,|x|\,w^*$$

$$\mathrm{Re}(x)_+ = e\,\mathrm{Re}(x)\,e$$

$$= \frac{1}{2}e(u|x| + |x|u^*)e$$

$$w|x|w^* - \mathrm{Re}(x)_+ \geq a|x|a^* - 2^{-1}e(u|x| + |x|u^*)e$$

$$= 4^{-1}e(1+u)|x|(1+u^*)e$$

$$-2^{-1}e(u|x| + |x|u^*)e$$

$$= 4^{-1}e\Big((1+u)|x|(1+u^*) - 2(u|x| + |x|u^*)\Big) \geq 0.$$

$|x| \mp |x|u^* \mp u|x| + u|x|u^*$

$(1-u)|x|(1-u^*)$

Lem 3.8 (Triangle ineq).

$x, y \in L^p(M)$ $0 < p \leq \infty$.

$\exists\, w_1, w_2 \in M^{PI}$

$$|x+y| \leq w_1|x|w_1^* + w_2|y|w_2^*.$$

pf

$x+y = v|x+y|$

No. $|x+y| = v^*(x+y)$

$= 2^{-1}(v^*(x+y) + (x+y)^* v)$

$= \mathrm{Re}\, v^* x + \mathrm{Re}\, v^* y$

$\leq (\mathrm{Re}\, v^* x)_+ + (\mathrm{Re}\, v^* y)_+$

$\leq w_1 |v^* x| w_1^* + w_2 |v^* y| w_1^*$ ∎

Lem 3.9

$x, y \in L^p(M)$, $0 < p \leq 1$.

$\|x+y\|_p^p \leq \|x\|_p^p + \|y\|_p^p$. q

$\leadsto \|x+y\|_p \leq 2^{1/p - 1}(\|x\|_p + \|y\|_p)$

~~quasi~~-norm

pf

$\|x+y\|_p^p = \mathrm{tr}(|x+y|^p)$

$\leq \mathrm{tr}((w_1|x|w_1^* + w_2|y|w_2^*)^p)$

Lem 3.6

$\leq \mathrm{tr}((w_1|x|w_1^*)^p + (\quad)^p)$

$= \|w_1|x|w_1^*\|_p^p + \_\_\_$. ∎

Lem 3.10

pf

$x, y \in L^p(M)_+$ $0 < p \leq 1$.

$\|x^p - y^p\|_1 \leq \|x-y\|_p^p$

pf. If $x \geq y \geq 0$.

$\|x-y\|_p^p - \|x^p - y^p\|_1 = \mathrm{tr}((x-y)^p)$

$- \mathrm{tr}(x^p - y^p)$

$= \mathrm{tr}((x-y)^p) - \mathrm{Tr}(x^p) + \mathrm{Tr}(y^p) \geq 0.$

$x, y \geq 0.$

$x - y = (x-y)_+ - (x-y)_-$

y $\quad x - y \leq x$

$\| x^p - y^p \|_1 \leq \| (y + (x-y)_+)^p - y^p \|_1$

$(x-y)_+ + y = x + (x-y)_-$

$+ \| (y + (x-y)_+)^p - x^p \|_1$

$\leq \| (x-y)_+ \|_p^p + \|(x-y)_-\|_p^p$

$= \| x - y \|_p^p .$

Pf of Kosaki Ineq : $\| |x| - |y| \| \leq 2^{1/2} \|x+y\|_1^{1/2} \|x-y\|_1^{1/2}$

Pmf - $\cancel{(\quad)}$

$$\boxed{\| |x| - |y| \|_1 \leq \| x^* x - y^* y \|_{1/2}^{1/2}}$$

By Lem 3.10

$x^* x - y^* y = 2^{-1} \left((x+y)^* (x-y) + (x-y)^* (x+y) \right)$

RHS $\leq 2^{-1/2} \| (x+y)^* (x-y) \|_{1/2}^{1/2} + 2^{-1/2} \| \quad \|_{1/2}^{1/2}$

$\leq 2^{-1/2} \|x+y\|_1 \|x-y\|_1^{1/2} + 2^{-1/2} \| \quad \|_1 \| \quad \|_1$

$= 2^{1/2} \sqrt{\|x+y\|_1 \|x-y\|_1} .$

Section 4 おまけ (植田さん)

M vN alg

φ faithful normal state

$M \subset B(H_\varphi)$ ← GNS Hilb sp.

$\varphi(x) = \langle x\xi_\varphi, \xi_\varphi \rangle$ ← GNS cyclic sep vector.

$[M\xi_\varphi] = H_\varphi = [M'\xi_\varphi]$

<u>Operators S & F</u>

$D(S_0) := M\xi_\varphi \quad S_0 x\xi_\varphi = x^*\xi_\varphi$

$D(F_0) := M'\xi_\varphi. \quad F_0 x'\xi_\varphi = x'^*\xi_\varphi.$

S_0 & F_0 : densely defn. conj lin. map

☆ $S_0 \subset F_0^*$, $F_0 \subset S_0^*$

In particular S_0 & F_0 closable.

$x\xi_\varphi \in D(S_0)$, $y'\xi_\varphi \in D(F_0)$

$$\begin{aligned}\langle S_0 x\xi_\varphi, y'\xi_\varphi \rangle &= \langle x^*\xi_\varphi, y'\xi_\varphi \rangle \\ &= \langle y'^* x^* \xi_\varphi, \xi_\varphi \rangle \\ &= \langle y'^*\xi_\varphi, x\xi_\varphi \rangle \\ &= \langle F_0 y'\xi_\varphi, x\xi_\varphi \rangle\end{aligned}$$

Let $S := \overline{S_0}$

$F := \overline{F_0}$.

<u>Lem 4.1</u> $\boxed{S = F^*, \quad F = S^*.}$

- $S \subset F^*$ O.K. since $S_0 \subset F_0^* = F^*$.

- Let $\xi \in D(F^*)$.

Define A_0 as

$$D(A_0) := M'\xi_\varphi$$

$$A_0 x'\xi_\varphi := x'\xi$$

▷ A_0 closable

$$\begin{cases} x_n'\xi_\varphi \longrightarrow 0 \\ A_0 x_n'\xi_\varphi \longrightarrow \eta \end{cases}$$

$\forall y' \in M'$

$$\begin{aligned}\langle \eta, y'\xi_\varphi \rangle &= \lim_n \langle A_0 x_n'\xi_\varphi, y'\xi_\varphi \rangle \\ &= \lim_n \langle x_n'\xi, y'\xi_\varphi \rangle \\ &= \lim_n \langle \xi, x_n'^* y'\xi_\varphi \rangle \\ &= \lim_n \langle \xi, F y'^* x_n'\xi_\varphi \rangle \\ &= \lim_n \langle y'^* \underbrace{x_n'\xi_\varphi}_{\downarrow 0}, F^*\xi \rangle \\ &= 0.\end{aligned}$$

$\leadsto \eta = 0.$

▷ $A_0 \eta M$ $\quad y'A_0 \subset A_0 y'$

Let $A := \overline{A_0}$

By defn

$$A\xi_\varphi = \xi \qquad A \eta M.$$

$$A^* \xi_\varphi = F^*\xi$$

$$\begin{aligned}\langle A x'\xi_\varphi, \xi_\varphi \rangle &= \langle x'\xi, \xi_\varphi \rangle \\ &= \langle \xi, x'^*\xi_\varphi \rangle \\ &= \langle \xi, F x'\xi_\varphi \rangle \\ &= \langle x'\xi_\varphi, F^*\xi \rangle \qquad \forall x' \in M'\end{aligned}$$

Then let $A = v|A|$ p.d.

$$\underset{M}{\overset{\cap}{v}} \quad |A| \,\eta\, M$$

$$|A| = \int_{[0,\infty)} \lambda \, de(\lambda)$$

$$A_n := v \int_{[0,n]} \lambda \, de(\lambda) \in M$$

$\rightsquigarrow$ $\forall \xi \in D(A)$ $\quad A_n \xi \to A\xi$.

Now we will show. $\xi \in D(F^*)$ is contained in $D(S)$.

$$A_n \xi_\varphi \to A\xi_\varphi = \xi$$

$$\begin{array}{c} S(A_n \xi_\varphi) \\ \| \\ A_n^* \xi_\varphi \end{array} \longrightarrow A^* \xi_\varphi = F^* \xi$$

Thus $\xi \in D(S)$ & $S\xi = F^*\xi$. ∎

☆ In this proof. we have found.

$\forall \xi \in D(S)$ $\exists A \,\eta\, M$ s.t. $\xi_\varphi \in D(A) \cap D(A^*)$

$$\begin{cases} \xi = A\xi_\varphi \\ S\xi = A^*\xi_\varphi \end{cases}$$

☆ Polar decompositions.

$$S = J_\varphi \Delta_\varphi^{1/2}$$

$$F = J_\varphi \Delta_\varphi^{-1/2}$$

they are derived from $S_0^2 \subset 1_{H_\varphi}$

$$J_\varphi^2 = 1 \qquad J_\varphi \Delta_\varphi J_\varphi = \Delta_\varphi^{-1}$$

J_φ : conj unitary

<u>Thm</u> 3.2 (Tomita)

$$\boxed{J_\varphi M J_\varphi = M' \qquad \Delta_\varphi^{it} M \Delta_\varphi^{-it} = M}$$

In the following we simply write $\Delta = \Delta_\varphi$
$J = J_\varphi$

When $\varphi = \tau$ tracial.

$$S x \xi_\tau = x^* \xi_\tau$$

is isometric $\leadsto$ $\Delta = 1$ $\leadsto$ $S = J = F$

$\leadsto$ ξ_τ tracial vector for M & M'.

Then easy to see

$$JMJ \subset M'$$

By symmetry

$$JM'J \subset M'' = M \text{ . done}$$

This deduces

$$M\xi_\tau = M'\xi_\tau$$

Also from this we can show

$$JMJ = M'$$

Indeed, let $x \in M$, $x' \in M'$ with

$$x^* \xi_\tau = x' \xi_\tau$$

Then $JxJ = x'$ since

$$J \underset{\underset{M}{\uparrow}}{x} J \overset{\overset{M}{\downarrow}}{y} \xi_\tau = J x y^* \xi_\tau$$
$$= y x^* \xi_\tau$$
$$= y x' \xi_\tau$$
$$= x' y \xi_\tau$$

$\leadsto$ $JMJ \subset M'$.

Need to treat the resolvent function.

No.

Lem 3.3

> Let $\lambda \in \mathbb{C}$, $-\lambda \in \mathbb{R}_+$
> &
> $a' \in M'$.
>
> Then $\exists\, a_\lambda \in M$ s.t.
>
> - $a_\lambda^* \xi_\varphi = (\Delta + \lambda)^{-1} a' \xi_\varphi$
> - $\| a_\lambda \| \leqq (2|\lambda| + \lambda + \bar{\lambda})^{-1} \| a' \|$

Proof.

Put $D(T_0) := M' \xi_\varphi$

$$T_0 x' \xi_\varphi := x' \underbrace{(\Delta + \lambda)^{-1} a' \xi_\varphi}_{\in D(\Delta)}$$

We know T_0 closable & $T_0 \,\eta\, M$.

So, we need to ~~compute~~ estimate $\| T_0 \|$.

Put $\xi := (\Delta + \lambda)^{-1} a' \xi_\varphi \in D(\Delta)$

Since $(\Delta + \bar{\lambda})^{-1} x'^* x' \xi \in D(\Delta)$

$\exists\, B \,\eta\, M$ closed. densely defined.

s.t.

- $\xi_\varphi \in D(B) \cap D(B^*)$
- $B \xi_\varphi = (\Delta + \bar{\lambda})^{-1} x'^* x' \xi_\varphi$

Then

$$\begin{aligned}
\| T x' \xi_\varphi \|^2 &= \| x' \xi \|^2 \\
&= \langle x'^* x' \xi , \xi \rangle \\
&= \langle (\Delta + \bar{\lambda}) B \xi_\varphi , (\Delta + \lambda)^{-1} a' \xi_\varphi \rangle \\
&= \langle B \xi_\varphi , a' \xi_\varphi \rangle \\
B = u|B| \qquad &= \langle |B|^{1/2} \xi_\varphi , |B|^{1/2} u^* a' \xi_\varphi \rangle \\
&= \langle |B|^{1/2} \xi_\varphi , a' |B|^{1/2} u^* \xi_\varphi \rangle \\
&\leqq \langle |B| \xi_\varphi , \xi_\varphi \rangle^{1/2} \langle u |B| u^* \xi_\varphi , \xi_\varphi \rangle^{1/2} \| a' \|
\end{aligned}$$

Now

$$\lambda\langle |B|\xi_\varphi, \xi_\varphi\rangle + \langle u|B|u^*\xi_\varphi, \xi_\varphi\rangle$$

$$= \lambda\langle u\xi_\varphi, B\xi_\varphi\rangle + \underset{\underset{\langle SB\xi_\varphi,\, Su\xi_\varphi\rangle}{\parallel}}{\langle B^*\xi_\varphi, u^*\xi_\varphi\rangle}$$

$$= \lambda\langle u\xi_\varphi, B\xi_\varphi\rangle + \langle u\xi_\varphi, \Delta B\xi_\varphi\rangle$$

$$= \langle u\xi_\varphi, (\Delta+\bar\lambda)B\xi_\varphi\rangle$$

$$= \langle \underset{\underset{M}{\uparrow}}{u}\xi_\varphi, \underset{\underset{M'}{\uparrow}}{x'^*} x'\xi_\varphi\rangle$$

$$= \langle u x'\xi_\varphi, x'\xi\rangle$$

$$\to \left|\lambda\langle |B|\xi_\varphi, \xi_\varphi\rangle + \langle u|B|u^*\xi_\varphi, \xi_\varphi\rangle\right|$$

$$\leqq \|x'\xi_\varphi\| \, \|x'\xi\| \qquad \text{— ②}$$

✡ $\alpha, \beta \in \mathbb{R}_+$. $\quad |\alpha + \lambda\beta| \leqq C$

$$\Rightarrow |\alpha\beta|^{1/2} \leqq (2|\lambda| + \lambda + \bar\lambda)^{-1/2} C$$

$$\left[\begin{aligned} C^2 &\geqq |\alpha+\lambda\beta|^2 - (\alpha - |\lambda|\beta)^2 \\ &= (\lambda+\bar\lambda)\alpha\beta + 2|\lambda|\alpha\beta \end{aligned}\right]$$

② $\to$

プラス!

$$\cancel{\lambda}\langle |B|\xi_\varphi, \xi_\varphi\rangle^{1/2} \langle u|B|u^*\xi_\varphi, \xi_\varphi\rangle^{1/2}$$

$$\leqq (2|\lambda| + \lambda + \bar\lambda)^{-1/2} \|x'\xi_\varphi\| \, \|x'\xi\| \qquad \text{— ③}$$

① & ③

$$\to \underset{\underset{\|x'\xi\|^2}{\parallel}}{\|Tx'\xi_\varphi\|^2} \leqq \|a'\| \, (2|\lambda| + \lambda + \bar\lambda)^{-1/2} \|x'\xi_\varphi\| \, \|x'\xi\|$$

$$\to \|x'\xi\| \leqq \|a'\| \, (2|\lambda| + \lambda + \bar\lambda)^{-1/2} \|x'\xi_\varphi\|$$

Lem 3.4

$\lambda \in \mathbb{C}$, $\lambda \notin -\mathbb{R}_+$

$a' \in M'$ given

Let $a_\lambda \in M$

$$a_\lambda^* \xi_\varphi = (\Delta + \lambda)^{-1} a' \xi_\varphi .$$

Then
$$J a' J = \Delta^{-1/2} a_\lambda \Delta^{1/2} + \bar{\lambda} \Delta^{1/2} a_\lambda \Delta^{-1/2}$$

on $D(\Delta^{1/2}) \cap D(\Delta^{-1/2})$ 　内積とったら等式

Proof.

$b', c' \in M'$. Take $b, c \in M$ s.t

$$b^* \xi_\varphi = (\Delta + 1)^{-1} b' \xi_\varphi$$

$$c^* \xi_\varphi = (\Delta + 1)^{-1} c' \xi_\varphi$$

Since $(\Delta + \lambda) a_\lambda^* \xi_\varphi = a' \xi_\varphi$,

$$\langle \Delta a_\lambda^* \xi_\varphi, b^* c \xi_\varphi \rangle + \lambda \langle a_\lambda^* \xi_\varphi, b^* c \xi_\varphi \rangle$$
$$= \langle a' \xi_\varphi, b^* c \xi_\varphi \rangle \quad \text{— ①}$$

The 1st term of ①

$= \langle \Delta a_\lambda^* \xi_\varphi, b^* c \xi_\varphi \rangle$

$= \langle FS a_\lambda^* \xi_\varphi, b^* c \xi_\varphi \rangle$

$= \langle S b^* c \xi_\varphi, S a_\lambda^* \xi_\varphi \rangle$

$= \langle c^* b \xi_\varphi, a_\lambda \xi_\varphi \rangle$

$= \langle b \xi_\varphi, c a_\lambda \xi_\varphi \rangle$

$= \langle S b^* \xi_\varphi, S a_\lambda^* c^* \xi_\varphi \rangle$

$= \langle \Delta a_\lambda^* c^* \xi_\varphi, b^* \xi_\varphi \rangle$

$= \langle \Delta a_\lambda^* (\Delta + 1)^{-1} c' \xi_\varphi, (\Delta + 1)^{-1} b' \xi_\varphi \rangle$

$= \langle \overset{\text{OK}}{\cancel{c' \xi_\varphi, (\Delta+1)}}^{-1} a_\lambda \Delta (\Delta + 1)^{-1} b' \xi_\varphi \rangle$ — 1st.

$\cancel{(\Delta+1)^{-1}\Delta}$

The 2nd term of ② $/\lambda$

$$= \langle a_\lambda^* \xi_\varphi,\ b^* c\, \xi_\varphi \rangle$$

$$= \langle b\, a_\lambda^* \xi_\varphi,\ c\, \xi_\varphi \rangle$$

$$= \langle S\, a_\lambda b^* \xi_\varphi,\ S\, c^* \xi_\varphi \rangle$$

$$= \langle \Delta c^* \xi_\varphi,\ a_\lambda b^* \xi_\varphi \rangle$$

$$= \langle c' \xi_\varphi,\ (\Delta+1)^{-1} \Delta\, a_\lambda (\Delta+1)^{-1} b' \xi_\varphi \rangle$$

The right-hand-side

$$= \langle a' \xi_\varphi,\ b^* c\, \xi_\varphi \rangle$$

$$= \langle a' b\, \xi_\varphi,\ c\, \xi_\varphi \rangle$$

$$= \langle a' S b^* \xi_\varphi,\ S\, c^* \xi_\varphi \rangle$$

$$= \langle c^* \xi_\varphi,\ F a' S b^* \xi_\varphi \rangle$$

$$= \langle c' \xi_\varphi, (\Delta+1)^{-1} \Delta^{1/2} J a' J \Delta^{1/2} (\Delta+1)^{-1} b' \xi_\varphi \rangle.$$

$$\rightsquigarrow (\Delta+1)^{-1} a_\lambda \Delta (\Delta+1)^{-1} + \bar{\lambda}\, (\Delta+1)^{-1} \Delta A_\lambda (\Delta+1)^{-1}$$

$$= (\Delta+1)^{-1} \Delta^{1/2} J a' J \Delta^{1/2} (\Delta+1)^{-1}$$

Proof of Thm 3.2.

Let $\xi, \eta \in H_\varphi$ analytic w.r.t. Δ

Put $\lambda > 0$.

$$f_\lambda(z) := \lambda^{-iz-1/2} \langle a_\lambda \Delta^{iz+1/2}\xi, \Delta^{i\bar{z}-1/2}\eta \rangle$$

$z \in \mathbb{C}$.

▷ $f_\lambda(z)$ bdd & hol on nbd of

$\{ z \in \mathbb{C} \mid 0 \le \mathrm{Im}\, z \le 1 \}$

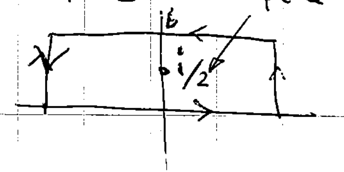
$z = x + iy$ で計算 $\to$ easy.

▷ Poisson integral formula.

$$f(i/2) = \int_{-\infty}^{\infty} \left(f(t) + f(t+i) \right) \frac{dt}{2\cosh(\pi t)}$$

Pf. Apply residue formula for

$g(z) := \dfrac{f(z)}{2\cos(\pi z)}$



Then $f_\lambda(i/2) = \langle a_\lambda \xi, \eta \rangle$

$$= \int_{-\infty}^{\infty} \lambda^{-it-1/2} \langle a_\lambda \Delta^{it+1/2}\xi, \Delta^{it-1/2}\eta \rangle \frac{dt}{2\cosh(\pi t)}$$

$$+ \int_{-\infty}^{\infty} \lambda^{-it+1/2} \langle a_\lambda \Delta^{it-1/2}\xi, \Delta^{it+1/2}\eta \rangle \frac{dt}{2\cosh(\pi t)}$$

$$\overset{\text{Lem 3.4}}{=} \int_{-\infty}^{\infty} \lambda^{-it-1/2} \langle \overset{\Delta^{-it}}{J} a' \overset{\Delta^{it}}{J} \xi, \eta \rangle \frac{dt}{2\cosh(\pi t)}$$

$$\rightsquigarrow \underset{M}{\overset{\Uparrow}{a_\lambda}} = \int_{-\infty}^{\infty} \lambda^{-it-1/2} \Delta^{-it} J a' J \Delta^{it} \frac{dt}{2\cosh(\pi t)}$$

Fourier transf は 単射. $\lambda = e^s \quad \forall s \in \mathbb{R}$

$\rightsquigarrow \Delta^{-it} J a' J \Delta^{it} \in M'' = M$

$\xrightarrow{t=0} J a' J \in M$. $\rightsquigarrow J M' J \subset M$

by symmetry

$JMJ \subset M'$

$\rightsquigarrow \Delta^{it} M \Delta^{-it} = M$