C^*-tensor category

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Section 1. C^*-tensor Categories

§1.1 Defn.

Defn 1.1

$\mathcal{C}$: category

is a <u>C^*-category</u>

if (1).(2) holds :

(1) $\mathrm{Mor}(\alpha, \beta)$ Banach space. for $\forall \alpha, \beta \in \mathcal{C}$

arrow

& For composition

$\alpha \xrightarrow{S} \beta \xrightarrow{T} \gamma$,

$\|TS\| \leq \|T\| \|S\|$.

(2) $\exists$ $*$-operation $\mathrm{Mor}(\alpha,\beta) \longrightarrow \mathrm{Mor}(\beta,\alpha)$

$T \longmapsto T^*$

conjugate linear

s.t.

(i) $T^{**} = T$

(ii) $(TS)^* = S^* T^*$ for $\alpha \xrightarrow{\forall S} \beta \xrightarrow{\forall T} \gamma$

(iii) $\|T^*T\| = \|T\|^2$ for $\alpha \xrightarrow{\forall T} \beta$

In particular, $\mathrm{End}(\alpha) := \mathrm{Mor}(\alpha, \alpha)$ is a C^*-alg for $\forall \alpha \in \mathcal{C}$

(iv) $T^*T \in \mathrm{End}(\alpha)_+$ for $\forall \alpha \xrightarrow{T} \forall \beta$

☆

(iv) is sometimes dropped. (automatically holds from the additional properties on ~~tensor~~ directsum structure.

Defn. 1.2.

$\mathcal{C}$: C^*-category

is a <u>C^*-tensor category</u>

if $\mathcal{C}$ is equipped with $(\otimes, a, l, r, \mathbb{1})$

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ bifunctor

  $(\alpha, \beta) \longmapsto \alpha \otimes \beta$, $\alpha, \beta \in \mathcal{C}$

  "binary operation"

  $\alpha \xrightarrow{S} \beta$, $\gamma \xrightarrow{T} \delta$

  $\leadsto S \otimes T \in \mathrm{Mor}(\alpha \otimes \gamma, \beta \otimes \delta)$

  bilinear w.r.t. S, T

  $(S \otimes T) \cdot (S' \otimes T') = SS' \otimes TT'$

- $a_{\alpha,\beta,\gamma} : (\alpha \otimes \beta) \otimes \delta \longrightarrow \alpha \otimes (\beta \otimes \delta)$

  natural unitary isomo. $\forall \alpha, \beta, \gamma \in \mathcal{C}$

  w.r.t. α, β & γ.

- $r_\alpha : \alpha \otimes \mathbb{1} \longrightarrow \alpha$

  $l_\alpha : \mathbb{1} \otimes \alpha \longrightarrow \alpha$ natural unitary isomos.

s.t. they are satisfying

(1) (Pentagon diagram)

$$\begin{tikzcd}[column sep=small]
 & ((\alpha \otimes \beta) \otimes \delta) \otimes \gamma \arrow[dl, "a_{\alpha,\beta,\gamma} \otimes 1_\gamma"'] \arrow[dr, "a_{\alpha \otimes \beta, \delta, \gamma}"] & \\
(\alpha \otimes (\beta \otimes \gamma)) \otimes \delta \arrow[d, "a_{\alpha, \beta\otimes\gamma, \delta}"'] & \circlearrowright & (\alpha \otimes \beta) \otimes (\delta \otimes \gamma) \arrow[d, "a_{\alpha,\beta,\delta\otimes\gamma}"] \\
\alpha \otimes ((\beta \otimes \gamma) \otimes \delta) \arrow[rr, "1_\alpha \otimes a_{\beta,\gamma,\delta}"] & & \alpha \otimes (\beta \otimes (\delta \otimes \gamma))
\end{tikzcd}$$

No.

(2) $l_{\mathbb{1}} = r_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$, and

$$\begin{tikzcd}[column sep=large] (\alpha \otimes \mathbb{1}) \otimes \beta \arrow[rr, "a_{\alpha,\mathbb{1},\beta}"] \arrow[dr, "r_\alpha \otimes 1_\beta"'] & & \alpha \otimes (\mathbb{1} \otimes \beta) \arrow[dl, "1_\alpha \otimes l_\beta"] \\ & \alpha \otimes \beta & \end{tikzcd}$$

(3) $(S \otimes T)^* = S^* \otimes T^*$ for $\forall S, T$ mors.

(4) (Direct sum).

For $\forall \alpha_1, \ldots, \alpha_n \in \mathcal{C}$

$\exists \beta \in \mathcal{C}$

$\exists u_i : \alpha_i \to \beta$ isometries ($i = 1, \ldots n$)

s.t.

$u_1 u_1^* + \cdots + u_n u_n^* = 1$

(5) (Subobject)

$\forall \alpha \in \mathcal{C}$ $\forall p \in \mathrm{End}(\alpha)$

$\exists \beta \in \mathcal{C}$ $\exists v : \beta \to \alpha$ isometry

s.t. $v v^* = p$

(6) $\mathrm{End}(\mathbb{1}) = \mathbb{C}1$;

(7) $\mathrm{Ob}(\mathcal{C})$ is a set!

☆ Naturality について.

$l_\alpha : \mathbb{1} \otimes \alpha \to \alpha$ is natural w.r.t. α

$\forall \alpha \xrightarrow{S} \beta$ $\quad S \circ l_\alpha = l_\beta (1 \otimes S)$

$$\begin{tikzcd} \mathbb{1} \otimes \alpha \arrow[r, "1 \otimes S"] \arrow[d, "l_\alpha"] & \mathbb{1} \otimes \beta \arrow[d, "l_\beta"] \\ \alpha \arrow[r, "S"] & \beta \end{tikzcd}$$

etc.

☆ $\alpha \in \mathcal{C}$ is <u>simple</u> or <u>irreducible</u>

$\overset{\text{defn}}{\Longleftrightarrow}$ $\mathrm{End}(\alpha) = \mathbb{C}1_\alpha$

ι_α とも書く.

$\mathrm{Irr}\,\mathcal{C} := \{\alpha \in \mathcal{C} \mid \alpha \text{ simple}\}$

☆ If

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

$$\mathbb{1} \otimes \alpha = \alpha = \alpha \otimes \mathbb{1} \qquad \forall \alpha, \beta, \gamma$$

then $\mathcal{C}$ is <u>strict</u>.

☆ (MacLane) Any $\mathcal{C}$ can be strictified.

($\exists \mathcal{D}$: strict C^*-tensor category

s.t. $\mathcal{C} \cong \mathcal{D}$ as a C^*-tensor cat.)

So, we always assume $\mathcal{C}$ is <u>strict</u>.

<u>Ex 1.3</u> $\mathrm{Hilb}_f = \{\text{fin. dim Hilb. spaces}\}$

$\otimes$ usual one. $\rightarrow$ strict C^*-tensor cat

we consider.

<u>Ex 1.4</u> G : compact quantum group

$\mathrm{Rep}\,G := \{(\pi, H_\pi) \mid \pi : G \to B(H)\}$

unitary repn

$(\pi \otimes \rho)(g) = \pi(g) \otimes \rho(g)$ $\rightarrow$ strict C^*-tensor cat

$\mathrm{Mor}(\pi, \rho) := \{T : H_\pi \to H_\rho \mid T\pi(g) = \rho(g)T\}$

<u>Ex 1.5</u> $\mathcal{TL}_{d,\tau}$ Temperley–Lieb category

$(d \geq 2)$

$d \in \{2\cos\frac{\pi}{k} \mid k = 3, 4, \ldots\} \cup [2, \infty)$ (reduced if $d < 2$)

$\tau \in \{\pm 1\}$

Ex 1.6 A : C^*-alg purely infinite.

$\mathrm{End}(A) := \{ \rho \mid \rho : A \to A \ *\text{-homo} \}$

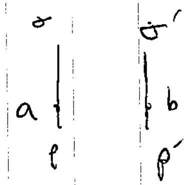
$\rho \otimes \sigma := \rho \circ \sigma$

$\mathrm{Mor}(\rho, \sigma) := \{ a \in A \mid a\rho(x) = \sigma(x) a \}$

$a : \rho \to \sigma$, $b : \rho' \to \sigma'$

$a \otimes b := \sigma(b) a = a \rho(b)$ $\qquad \rho \circ \rho' \to \sigma \circ \sigma'$

$\sigma(b) a \rho\rho'(x) = \sigma(b \rho'(x)) a = \sigma(\sigma'(x) b) a = \sigma\sigma'(x) \sigma(b) a$



pure infiniteness

$\rightsquigarrow$ directsum & subobject

$\to$ strict C^*-tensor category

A properly infinite $\rightsquigarrow$ rigid.

Ex. 1.7

Γ discrete group

$\mathrm{Rep}(\hat{\Gamma}) := \{ \Gamma\text{-graded fin. dim Hilb. spaces} \}$

$= \{ (s, H) \mid s \in \Gamma,\ H \in \mathrm{Hilb}_f \}$

$\mathrm{Mor}((s, H), (t, K)) = \delta_{s,t}\, B(H, K)$.

$(s, H) \circ (t, K) := (st, H \circ K)$.

$\longrightarrow$ strict C^*-tensor category.

Often denoted like

$(s, H) = H_s$.

Ex. 1.8 $N \subset M \subset M_1$ subfactor, basic extension

$\rightsquigarrow {}_M L^2 M_{1\,M}$ bimodule $\subset {}_M\mathrm{Hilb}_M$

generates $\mathcal{C}_{N \subset M}$. $\qquad \otimes_M$ strict C^*-tensor cat

§1.2 Conjugate objects & intrinsic dimension

$\mathcal{C}$: strict C^*-tensor category

Defn. 1.9

$\alpha, \beta \in \mathcal{C}$

β is a conjugate object of α

if $\exists R : \mathbb{1} \longrightarrow \beta \otimes \alpha$

$\exists \bar{R} : \mathbb{1} \longrightarrow \alpha \otimes \beta$

s.t.

$$\left.\begin{aligned} (1_\alpha \otimes R^*) \circ (\bar{R} \otimes 1_\alpha) &= id_\alpha \\ (1_\beta \otimes \bar{R}^*) \circ (R \otimes 1_\beta) &= id_\beta \end{aligned}\right\} (*)$$

☆ $(*)$ を conjugate equation と呼ぶ.

☆ β is conj. obj. of α

$\Leftrightarrow$ α is conj. obj. of β

☆ Conjugate objects are unique up to unitary equivalence. Prop 1.13

often denoted by $\bar{\alpha}$.

Defn. 1.10

$\mathcal{C}$ is rigid

if $\forall \alpha \in \mathcal{C}$ has a conj. object.

Ex. 1.11

$H \in \mathrm{Hilb}_f$

$\bar{H} := \{ \bar{\xi} \mid \xi \in H \}$ conj Hilb space.

$$\begin{cases} \overline{\xi + \eta} = \bar{\xi} + \bar{\eta}, & \langle \bar{\xi}, \bar{\eta} \rangle_{\bar{H}} := \langle \eta, \xi \rangle_H \\ \overline{\lambda \xi} = \bar{\lambda} \bar{\xi} \end{cases}$$

e_i : ONB of H

$r : \mathbb{1} \to \overline{H} \otimes H$ $\qquad r(1) := \sum_{i=1}^{n} \overline{e_i} \otimes e_i$

$\mathbb{1} = \mathbb{C}1$

$\overline{r} : \mathbb{1} \to H \otimes \overline{H}$ $\qquad \overline{r}(1) := \sum_{i=1}^{n} e_i \otimes \overline{e_i}$

They do not depend on e_i's.

$\overline{H}$ conj. obj. of H.

$\rightsquigarrow$ Hilb_f rigid.

Ex. 1.12

Rep G

$\overline{(\pi, H)} := (\overline{\pi}, \overline{H})$

$\overline{\pi}(g)\,\overline{\xi} := \overline{\pi(g)\xi}$

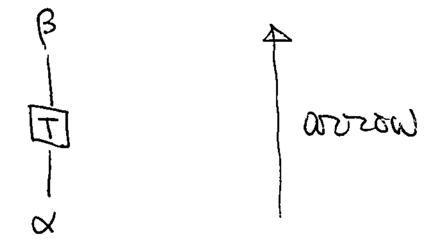
Graphical interpretation

In what follows, we will treat intertwiners with many $\otimes$'s. Graphical interpretation is quite useful to understand the positions of $\otimes$'s.

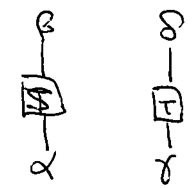
For example,

$T : \alpha \to \beta$

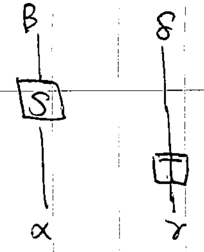
is like



$S : \alpha \to \beta \qquad T : \gamma \to \delta$

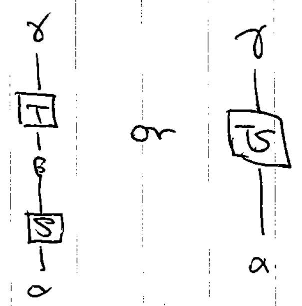
$S \otimes T =$ 

that is equal to $(S \otimes 1_\delta) \cdot (1_\alpha \otimes T)$

= 

The composition is like

$S : \alpha \longrightarrow \beta$, $T : \beta \rightarrow \gamma$

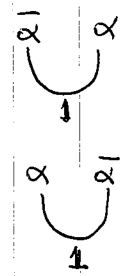
$T \circ S =$ 

Let $\alpha \in \mathcal{C}$ & $\bar{\alpha}$: its conj.

$(R, \bar{R})$ solution of conj eq.

$R : \mathbb{1} \rightarrow \bar{\alpha} \otimes \alpha$

$\bar{R} : \mathbb{1} \rightarrow \alpha \otimes \bar{\alpha}$.

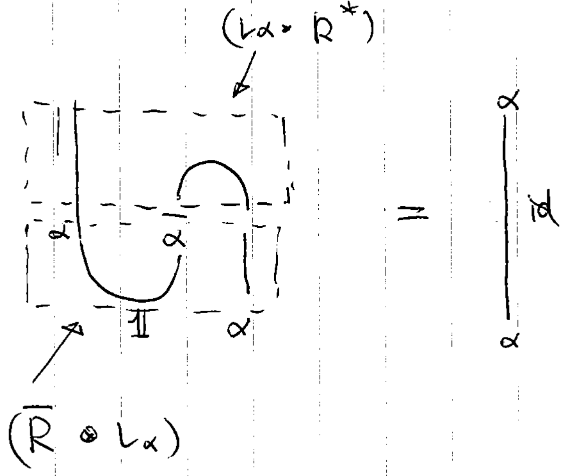


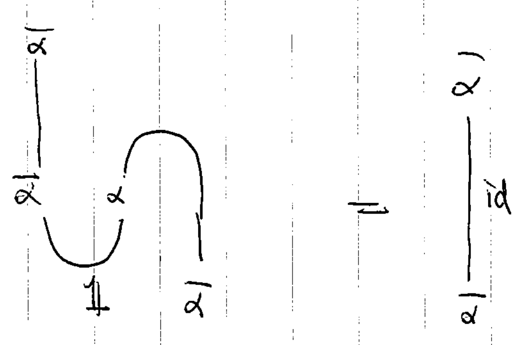
The equations

$(\iota_\alpha \otimes \bar{R}_\alpha^*) \circ (\bar{R} \otimes \iota_\alpha) = \iota_\alpha$

$(\iota_{\bar{\alpha}} \otimes \bar{R}^*) \circ (R \otimes \iota_{\bar{\alpha}}) = \iota_{\bar{\alpha}}$

are like



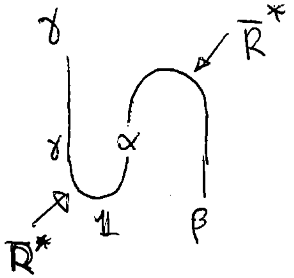


Prop. 1.12

> $\alpha \in \mathcal{C}$, β, γ は α の conj. obj とし、
> $(R, \bar{R}^*)$, $(R', \bar{R}')$ を conj. equation の解とすると、$\exists ! \, T : \beta \to \gamma$ invertible
> (unitary) s.t.
> $$R' = (T \otimes 1_\alpha) \circ R$$
> $$\bar{R}' = (1_\alpha \otimes T^{-1*}) \circ \bar{R}$$
> とくに $\beta \cong \gamma$ (unitary equiv).

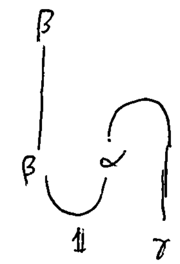
Proof.

$T := (1_\gamma \otimes \bar{R}^*) \circ (R' \otimes 1_\beta)$

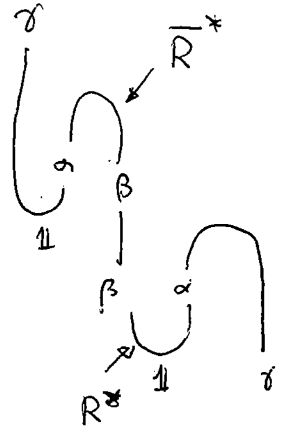
$=$ 

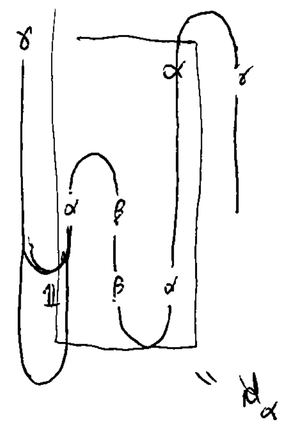
$\in \mathrm{Mor}(\beta, \gamma)$

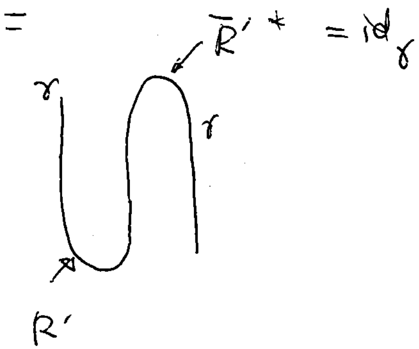
$S := (1_\beta \otimes \bar{R}'^*) \circ (R \otimes 1_\gamma)$

$=$ 

$\in \mathrm{Mor}(\gamma, \beta)$

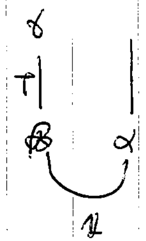
$TS =$ 

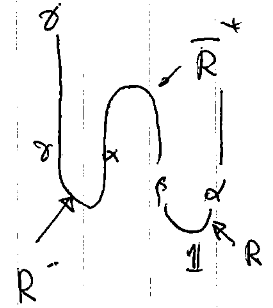
$=$ 

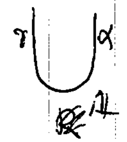
$=$ 

$ST = \mathrm{id}_\beta$

$\rightsquigarrow$ T, S are invertible

$(T \circ 1_\alpha) \cdot R =$ 

$=$ 

$=$  $= R'$

☆ So, ~~once~~ we fix ~~the~~ a conj. of α.

we often denote it by $\bar{\alpha}$.

Thm. 1.14 (Frobenius reciprocity)

Let $\alpha \in \mathcal{C}$ & $\bar{\alpha}$ its conj.

$(R_\alpha, \bar{R}_\alpha)$ sol. of conj. eq.

(1) $\mathrm{Mor}(\alpha \otimes \beta, \gamma) \xrightarrow[\sim]{\text{vector sp.}} \mathrm{Mor}(\beta, \bar{\alpha} \otimes \gamma)$

$T \longmapsto (1_{\bar{\alpha}} \otimes T)(R_\alpha \otimes 1_\beta)$

$(\bar{R}_\alpha^* \otimes 1_\gamma) \circ (1_\alpha \otimes S) \longleftarrow S$

(2) $\mathrm{Mor}(\beta \otimes \alpha, \gamma) \xrightarrow{\sim} \mathrm{Mor}(\beta, \gamma \otimes \bar{\alpha})$

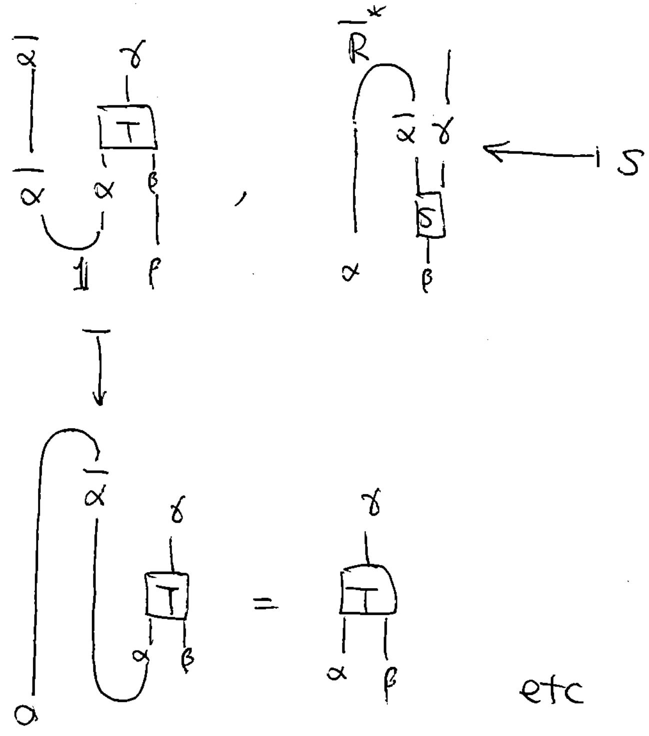
$T \longmapsto (T \otimes 1_{\bar{\alpha}})(1_\beta \otimes \bar{R}_\alpha)$

$(1_\gamma \otimes R_\alpha^*) \cdot (S \otimes 1_\alpha) \longleftarrow S$

Proof.

~~Proof~~

(1) $T \longmapsto$



etc ∎

Cor. 1.15

> If $\alpha \in \mathcal{C}$ simple & has conj. $\bar{\alpha}$
>
> then
>
> $$\mathrm{Mor}(1\!\!1, \bar{\alpha} \circ \alpha) = \mathbb{C} R$$
>
> $$\mathrm{Mor}(1\!\!1, \alpha \circ \bar{\alpha}) = \mathbb{C} \bar{R}$$
>
> one-dim.

Proof.

$$\mathrm{Mor}(1\!\!1, \bar{\alpha} \circ \alpha) \xrightarrow[\sim]{\text{Frob.}} \mathrm{Mor}(\alpha, \alpha) = \mathbb{C}\, 1_\alpha$$ ∎

Thm. 1.16

> If $\alpha \in \mathcal{C}$ has conj. obj,
>
> then $\mathrm{End}(\alpha)$ is fin. dim C^*-alg.

Proof.

Set
$$\begin{array}{cccc} \varphi_\alpha : & \mathrm{End}(\alpha) & \longrightarrow & \mathbb{C} \\ & \in & & \in \\ & T & \longmapsto & R^*(1_{\bar{\alpha}} \circ T) R \end{array}$$

Recall

$$\mathrm{End}(\alpha) \xrightarrow{\sim} \mathrm{Mor}(1\!\!1, \bar{\alpha} \circ \alpha)$$

$$\begin{array}{ccc} T & \longleftrightarrow & S \\ \| & & \| \\ (\bar{R}^* \circ 1_\alpha)(1_\alpha \circ S) & & (1_{\bar{\alpha}} \circ T) R \end{array}$$

Then

$$T^*T = (1_\alpha \otimes S^*)\cdot(\bar{R}\bar{R}^* \otimes 1_\alpha)\cdot(1_\alpha \otimes S)$$

$$\leqq \|\bar{R}\bar{R}^*\| \, (1_\alpha \otimes \underbrace{S^*S}_{= \varphi_\alpha(T^*T)}) \qquad \text{---}(*)$$

Hence $\varphi_\alpha(T) \geqq \dfrac{1}{\|\bar{R}\bar{R}^*\|} T$, $\forall T \in \mathrm{End}(\alpha)_+$.

$\rightsquigarrow$ $\dim \mathrm{End}(\alpha) < +\infty$. ▨

☆ If put $T = 1_\alpha$ in $(*)$, we have

$$1_\alpha \leqq \|\bar{R}\bar{R}^*\| \; R^*R$$
$$= \|\bar{R}\|^2 \|R\|^2 \, 1_\alpha$$

$\rightsquigarrow$ $\|R\|\|\bar{R}\| \geqq 1$.

$\mathrm{Mor}(\beta_k, \alpha)$

v_i^k ONB

$i = 1 \dots n_k$

$v_i^k v_j^{k*}$

☆ Irreducible decomposition.

$\alpha \in \mathcal{C}$, $\bar{\alpha}$: conj obj of α.

$$\mathrm{End}(\alpha) \simeq \bigoplus_{k=1}^{n} M_{n_k}(\mathbb{C})$$

$\{e_{ij}^k\}_{i,j}$ matrix units

$\beta_k := e_{11}^k \alpha \quad (k = 1, \dots n)$

(i.e. $\beta_k \xrightarrow{\exists v^k} \alpha$ s.t isometry $v_k v_k^* = e_{11}^k$)

$v_i^k := e_{i1}^k v^k : \beta_k \longrightarrow \alpha$ isom.

$v_i^k v_i^{k*} = e_{i1}^k e_{11}^k e_{1i}^k = e_{ii}^k$

$\Downarrow$

$$\sum_{k=1}^{n} \sum_{i=1}^{n_k} v_i^k v_i^{k*} = 1_\alpha$$

i.e. $\alpha \simeq \bigoplus_{k=1}^{n} \overbrace{\beta_k \oplus \cdots \oplus \beta_k}^{n_k \text{ times}}$

Each β_k is simple:

$$\mathrm{End}(\beta_k) \xrightarrow[\sim]{\mathrm{Ad}\, v_k} e^k_{11}\,\mathrm{End}(\alpha)\, e^k_{11} = \mathbb{C} e^k_{11}$$

& if $k \neq l$ then $\beta_k \not\cong \beta_l$

$$\begin{array}{ccc} \mathrm{Mor}(\beta_k, \beta_l) & \hookrightarrow & \mathrm{End}(\alpha) \\ \Psi & & \Psi \\ T & \longmapsto & v^l T v^{k*} \in 1_{n_l} \mathrm{End}(\alpha) 1_{n_k} \\ & & \quad\; \beta_l \leftarrow \beta_k \leftarrow \alpha \qquad \quad \| \\ & & \qquad\qquad\qquad\quad 0 \end{array}$$

So, α decomposes into direct sum of β_k's simple

This dec. is unique up to unitary eq.

Note Schur Lemma:

$\mathrm{Mor}(\beta, \gamma) \neq 0 \Rightarrow \beta \cong \gamma$ unitary
(β, γ simple)

Suppose

$$\overset{n}{\underset{k=1}{\oplus}} (\overbrace{\beta_k \oplus \cdots \oplus \beta_k}^{n_k}) \cong \alpha \cong \overset{m}{\underset{r=1}{\oplus}} (\overbrace{\gamma_r \oplus \cdots \oplus \gamma_r}^{m_r})$$

$\beta_k \not\cong \beta_l \quad k \neq l \qquad\qquad \gamma_r \not\cong \gamma_s \quad r \neq s$

$$\beta_k \xrightarrow{v^k_i} \alpha \qquad i = 1 \ldots n_k$$

$$\uparrow w^r_j \qquad j = 1, \cdots, m_r$$

$$\gamma_r$$

k fix

Since $\displaystyle\sum_{j,r} w^r_j w^{r*}_j = 1_\alpha$

$\exists$ At least one j, r

$$v^{k*}_i w^r_j \neq 0$$

$$\cap$$

$$\mathrm{Mor}(\gamma_r, \beta_k)$$

$\rightsquigarrow \beta_k \cong \gamma_r$

So, $n = m$. we may assume $\beta_k = \gamma_k$.

The multiplicity n_k is nothing but

the dim of $\mathrm{Mor}(\beta_k, \alpha)$ $\quad\lrcorner$

★ $\mathcal{C}$ C^*-tensor cat.

$\cup$

$\mathcal{C}_{conj}$: conj となる object たち

Mor-set は同じもの

$\rightarrow$ full subcategory

Infact $\mathcal{C}_{conj}$ is C^*-tensor cat.

i.e. closed under $\otimes$, direct sum, subobject.

$\underline{\otimes}$

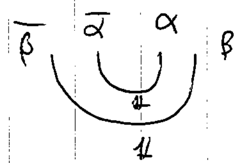
$\alpha \xleftrightarrow{\text{dual}} \bar{\alpha} \qquad (R_\alpha, \bar{R}_\alpha)$

$\beta \xleftrightarrow{\text{dual}} \bar{\beta} \qquad (R_\beta, \bar{R}_\beta)$

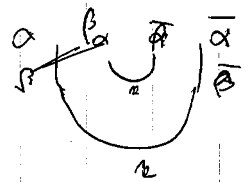
$\alpha \otimes \beta \xleftrightarrow{\text{dual}} \bar{\beta} \otimes \bar{\alpha}$ by $(R_{\alpha\cdot\beta}, \bar{R}_{\alpha\cdot\beta})$

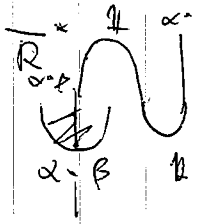
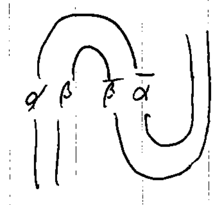
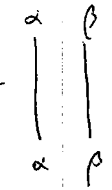
where

$$R_{\alpha\cdot\beta} := (1_{\bar{\beta}} \cdot R_\alpha \cdot 1_\beta) \circ R_\beta$$

$=$ 

$$\overline{R}_{\alpha\cdot\beta} = (1_\alpha \cdot \bar{R}_\beta \cdot 1_{\bar{\alpha}}) \circ \bar{R}_\alpha$$

$=$ 

 $R_{\alpha\cdot\beta}$ $=$  $=$  $= id_{\alpha\cdot\beta}$

etc.

Direct sum.

$$\gamma := \alpha \oplus \beta. \qquad \delta := \bar\alpha \oplus \bar\beta$$

$$v \uparrow_\alpha \quad \nwarrow^{w}_{\beta} \qquad s \uparrow_{\bar\alpha} \quad \nwarrow^{t}_{\bar\beta}$$

$$R_\gamma \doteq \mathbb{1} \to \delta \circ \gamma$$

$$R_\gamma := (s \circ v) \circ \underbrace{R_\alpha}_{\bar\alpha \circ \alpha \leftarrow \mathbb{1}} + (t \circ w) \circ \underbrace{R_\beta}_{\bar\beta \circ \beta \leftarrow \mathbb{1}}$$

$$\bar R_\gamma := (v \circ s) \circ \bar R_\alpha + (w \circ t) \cdot \bar R_\beta$$

Subobject.

$\alpha \,,\, \bar\alpha$ dual pair

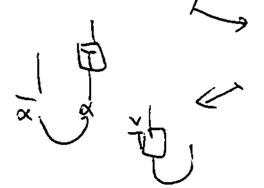
$$\begin{array}{ccc} \alpha & & \bar\alpha \\ v\uparrow & \dashrightarrow & \uparrow w? \\ \beta & & ? \end{array}$$

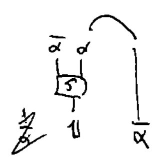
$\mathrm{Mor}(\alpha, \alpha)$

$$\mathrm{End}(\alpha) \longrightarrow \mathrm{Mor}(\mathbb{1}, \bar\alpha \circ \alpha) \longrightarrow \mathrm{Mor}(\bar\alpha, \bar\alpha)$$

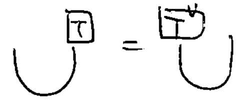
$$T \longmapsto (1 \circ T) \cdot R_\alpha \overset{S}{\longmapsto} (1_{\bar\alpha} \circ R_\alpha^*)(S \circ 1_{\bar\alpha})$$

$$\longmapsto T^\vee$$





$$(1_{\bar\alpha} \circ T) \cdot R_\alpha = (T^\vee \circ 1_\alpha) \cdot R_\alpha$$



$\to$ $T \longmapsto T^\vee$ anti multiplicative map

(not in general $*$-preserving)

If $(R, \bar R)$ std then $*$-preserving

$p := vv^* \in \mathrm{End}(\alpha)$

$\to$ $p^\vee \in \mathrm{End}(\bar\alpha)$ idempotent.

$\quad \|$

$t\, q\, t^{-1}$ $\quad q$ projection. $t \in \mathrm{End}(\bar\alpha)^\times$

$\quad \in \mathrm{End}(\bar\alpha)$

Take $\gamma \in \mathcal{C}$ s.t.

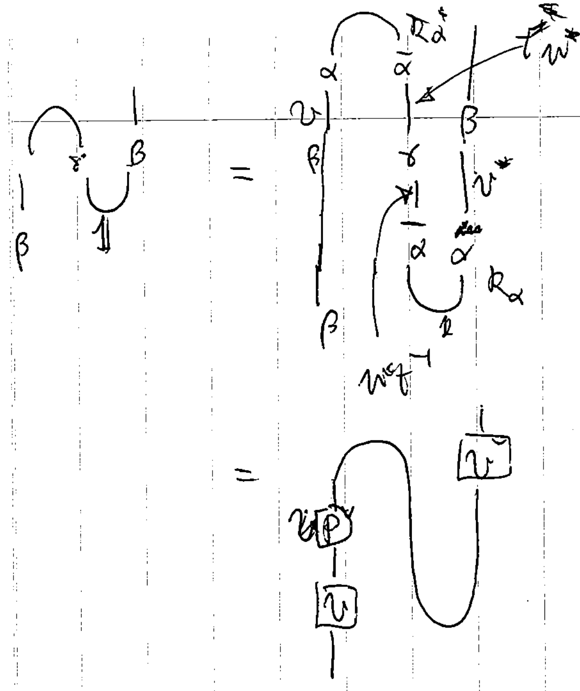
$$\gamma \xrightarrow[\text{isom}]{\ w\ } \bar\alpha \qquad ww^* = q$$

$$\bar R_\beta : \mathbb{1} \to \underset{\beta \circ \gamma}{\cancel{\gamma \circ \beta}} \qquad \bar R_\beta = (v^* \circ w^* t^*) \cdot \bar R_\alpha$$

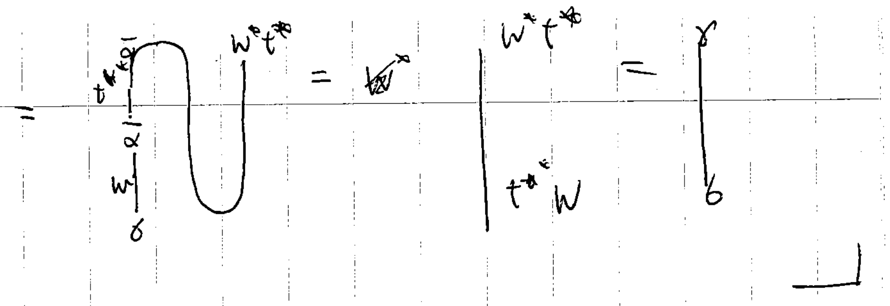
$$\bar{\bar R}_\beta : \mathbb{1} \to \underset{\gamma \circ \beta}{\cancel{\beta \circ \gamma}} \qquad \bar{\bar R}_\beta = (w^* t^{-1} \circ v^*)\, \bar{\bar R}_\alpha$$

$$S \longleftrightarrow tw$$

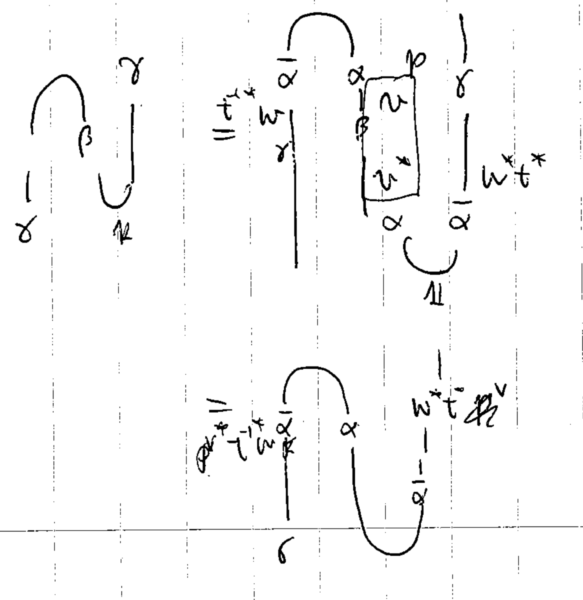
$$T \longleftrightarrow w^* t^{-1}$$



$$t^{-*} w w^{q*} t^{-1} = p^\vee$$



$$= \iota_\beta$$



$$p^{\vee *} t^{-1} w^{*} = (t^{-1} p^\vee)^* w = (q t^{-*})^* w = t^{-*} q w = \tilde{w}$$

Intrinsic dimension.

$\mathcal{C}$ rigid C^*-tensor cat.

$\alpha \underset{\mathrm{Irr}}{\in} \mathcal{C} \overset{\text{conj}}{\longleftrightarrow} \bar{\alpha} \underset{\mathrm{Irr}}{\in} \mathcal{C}$

$(R, \bar{R})$ sol. of conj. eq.

Other sols are

$$(\lambda R, \bar{\lambda}^{-1} \bar{R}), \quad \lambda \in \mathbb{C}^\times$$

$\rightsquigarrow \|\lambda R\| \, \|\bar{\lambda}^{-1} \bar{R}\| = \|R\| \, \|\bar{R}\|$

$$d(\alpha) := \|R\| \, \|\bar{R}\| \quad \text{int. dim. of } \alpha$$

$$\geqq 1 \qquad \| \; d(\bar{\alpha})$$

案れ $\|R_\alpha\| = \|\bar{R}_\alpha\| = d(\alpha)^{1/2}$ と取る.

In general, let $\alpha \in \mathcal{C}$.

$$\alpha = \bigoplus_{k=1}^{n} \alpha_k \quad \text{irred. dec.}$$

$$d(\alpha) := \sum_{k=1}^{n} d(\alpha_k)$$

☆ $d(\alpha \oplus \beta) = d(\alpha) + d(\beta)$

$d(\bar{\alpha}) = d(\alpha)$

since $\alpha = \oplus \alpha_k \Rightarrow \bar{\alpha} = \oplus \bar{\alpha}_k$

$d(1) = 1$

☆ Now put $\alpha_k \xrightarrow[\text{simple}]{v_k \text{ isoms}} \alpha \quad k = 1 \cdots n. \quad \sum v_k v_k^* = 1_\alpha$

$\bar{\alpha}_k \xrightarrow{w_k} \bar{\alpha} \qquad \sum w_k w_k^* = 1_{\bar{\alpha}}$

$$\left.\begin{aligned} R_\alpha &:= \sum_k (w_k \otimes v_k) \cdot R_{\alpha_k} \\ \bar{R}_\alpha &:= \sum (v_k \otimes w_k) \cdot \bar{R}_{\alpha_k} \end{aligned}\right\} (*) \text{ model} \qquad \|R_{\alpha_k}\| = d(\alpha_k)^{1/2} = \|\bar{R}_{\alpha_k}\|$$

we call model solution

Then $(R_\alpha, \bar{R}_\alpha)$ sol. of conj eq. w.r.t. $(\alpha, \bar{\alpha})$

& $R_\alpha^* R_\alpha = \sum_k R_{\alpha_k}^* R_{\alpha_k} = \sum_k \underbrace{\|R_{\alpha_k}\|^2}_{d(\alpha_k)} = d(\alpha)$

$\bar{R}_\alpha^* \bar{R}_\alpha = d(\alpha) \rightsquigarrow d(\alpha) = \|R_\alpha\| \, \|\bar{R}_\alpha\|$.

☆ For a sol. of conj eq. $(R, \bar{R})$ for α & $\bar{\alpha}$.

$$\varphi_\alpha^{(R,\bar{R})} : \mathrm{End}(\alpha) \longrightarrow \mathbb{C}$$
$$\quad\ \ \in \qquad\qquad\quad \in$$
$$T \longmapsto R^*(1_{\bar{\alpha}} \cdot T) R$$

$$\psi_\alpha^{(R,\bar{R})} : \mathrm{End}(\alpha) \longrightarrow \mathbb{C}$$
$$\in$$
$$T \longmapsto \bar{R}^*(T \cdot 1_{\bar{\alpha}}) \bar{R}$$

positive functionals $\varphi_\alpha(1_\alpha) = \|R\|^2$

$\psi_\alpha(1_{\bar{\alpha}}) = \|\bar{R}\|^2$

If we take $(R_\alpha, \bar{R}_\alpha)$ by $(*)$,

$$\varphi_\alpha(T) = \sum_k R_{\alpha_k}^* (1 \cdot v_k^* T v_k) R_{\alpha_k}$$

$$= \sum_k \varphi_{\alpha_k}(\underbrace{v_k^* T v_k}_{\substack{\cap \\ \alpha_k \to \alpha_k}})$$

$$= \sum_k (v_k^* T v_k) \cdot \varphi_{\alpha_k}(1_{\alpha_k})$$

scalar $\times 1_{\alpha_k}$ $\quad \|R_{\alpha_k}\|^2$.

$$\psi_\alpha(T) = \sum \bar{R}_{\alpha_k}^* (v_k^* T v_k \cdot 1) \bar{R}_{\alpha_k}$$

scalar $\times 1_{\alpha_k}$

$$= \sum (v_k^* T v_k) \cdot \|\bar{R}_{\alpha_k}\|^2$$

$\|R_{\alpha_k}\|^2 = d(\alpha_k)$

$$= \varphi_\alpha(T).$$

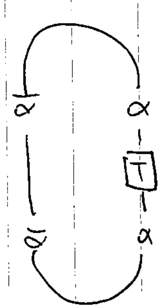
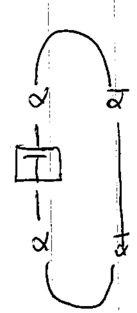
Moreover $\varphi_\alpha = \psi_\alpha$ is tracial. $\Big(\bigoplus_{i=1}^N M_{n_i}(\mathbb{C})$. $d(\alpha_k) \mathrm{Tr}_{M_{n_i}(\mathbb{C})}$

faithful.

<u>Defn</u>. 1. ~~16~~17

> $\alpha \in \mathcal{C}$.
>
> conj. eq. $(R, \bar{R})$ is <u>standard</u>
>
> iff $\varphi_\alpha^{(R,\bar{R})} = \psi_\alpha^{(R,\bar{R})}$

☆ $(R, \bar{R})$ std

$\overset{\text{dfn}}{\Longleftrightarrow}$  $=$  $\forall T \in \mathrm{End}(\alpha)$

☆ $(R_\alpha, \bar{R}_\alpha)$ by (*) is standard.

$(\bar{R}_\alpha, R_\alpha)$

☆ standard solution of conj eq. is unique:

Suppose $(R, \bar{R})$ std. sol. of conj eq.

$\alpha \leftrightarrow \bar{\alpha}$

Then $\exists\, T \in End(\alpha)$ invertible

s.t. $\bar{R} = (1_{\bar{\alpha}} \circ T)\bar{R}_\alpha$ (*).

$\bar{R} = (T^{*-1} \circ 1_{\alpha})\bar{R}_\alpha$ $= (T$

$R = (1_{\bar{\alpha}} \circ T) R_\alpha$

$\bar{R} = (T^{*-1} \circ 1_{\bar{\alpha}}) R_{\bar{\alpha}}$

$$\varphi^{(R,\bar{R})}(x) = \varphi_\alpha(T^* x T) \overset{\text{tracial}}{=} \varphi_\alpha(TT^* x)$$

$$\| \qquad\qquad\qquad\qquad \#$$

$$\psi^{(R,\bar{R})}(x) = \psi_\alpha(T^{-1} x T^{*-1}) = \psi_\alpha(\underset{\underset{(TT^*)^{-1}}{\|}}{T^{*-1}T^{-1}} x) = \varphi_\alpha((TT^*)^{-1} x)$$

$\rightsquigarrow$ $TT^* = (TT^*)^{-1}$. T unitary.

So, any std sol. is of the form (*).

☆ $Tr_\alpha : End(\alpha) \longrightarrow \mathbb{C}$

$Tr_\alpha(x) = R^*(1 \circ x)R$ std. sol.

$= \bar{R}^*(x \circ 1)\bar{R}$

faithful tracial positive functional.

$Tr_\alpha(1_\alpha) = d(\alpha)$.

$$\|R\| = \|\bar{R}\|$$

$$\| \qquad \|$$

$$\|R_\alpha\| \quad \|\bar{R}_\alpha\|$$

$\|$ model.

$d(\alpha)^{1/2}$

Thm. 1.18

$\alpha, \beta \in \mathcal{C}$.

$(R_\alpha, \bar{R}_\alpha)$, $(R_\beta, \bar{R}_\beta)$ std sols.

$\Rightarrow (R_{\alpha\otimes\beta}, \bar{R}_{\alpha\otimes\beta})$ std sols for $\alpha\otimes\beta$

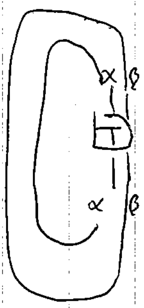
where

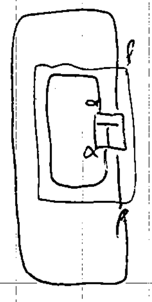
$R_{\alpha\otimes\beta} = (\iota_{\bar\beta} \otimes R_\alpha \otimes \iota_\beta) \cdot R_\beta$

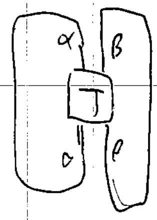
$\bar{R}_{\alpha\otimes\beta} = (\iota_\alpha \otimes \bar{R}_\beta \otimes \iota_{\bar\alpha}) \cdot \bar{R}_\alpha$

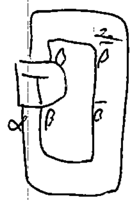
Proof.

Let $T \in \mathrm{End}(\alpha\otimes\beta)$

$\varphi^{(R_{\alpha\otimes\beta}, \bar{R}_{\alpha\otimes\beta})}(T) =$ 

$=$  $\in \mathrm{End}(\beta, \beta)$

$=$  stdness of $(R_\rho, \bar{R}_\rho)$

$=$  $= \psi^{(R_{\alpha\otimes\beta}, \bar{R}_{\alpha\otimes\beta})}(T)$.

☆ $\mathrm{Tr}_{\alpha\otimes\beta}(S\otimes T) = \mathrm{Tr}_\alpha(S)\,\mathrm{Tr}_\beta(T)$

Cor. 1.19

$d(\alpha\otimes\beta) = d(\alpha)\,d(\beta)$

Proof

$d(\alpha\otimes\beta) = \mathrm{Tr}_{\alpha\otimes\beta}(1_\alpha \otimes 1_\beta)$

$= \mathrm{Tr}_\alpha(1_\alpha)\,\mathrm{Tr}_\beta(1_\beta)$

$= d(\alpha)\,d(\beta)$

Thm. 1.~~19~~20

$\forall \alpha \in \mathcal{C}$

$$d(\alpha) = \min \{ \|R\| \|\bar{R}\| \mid (R, \bar{R}) \text{ sol. conj eq. for } \alpha \}$$

The minimum $(R, \bar{R})$ is a scalar mult of std. sol.

Proof.

$(R_\alpha, \bar{R}_\alpha)$ std. sol.

$\rightarrow d(\alpha) = \|R_\alpha\|^2 = \|\bar{R}_\alpha\|^2 \ldots$

$(R, \bar{R})$ othe sol. $\exists T \in \mathrm{End}(\alpha)$ inv.

$R = (1_{\bar{\alpha}} \otimes T) \cdot R_\alpha$

$\bar{R} = (T^{*-1} \otimes 1_{\bar{\alpha}}) \cdot R_{\bar{\alpha}}$

$R^* R = \mathrm{Tr}_\alpha(T^* T)$

$\bar{R}^* \bar{R} = \mathrm{Tr}_\alpha(T^{-1} T^{*-1})$

$\rightsquigarrow$

$$\|R\| \|\bar{R}\| = \mathrm{Tr}_\alpha(T^* T)^{\frac{1}{2}} \, \mathrm{Tr}_\alpha((T^* T)^{-1})^{\frac{1}{2}}$$

$$\geq \mathrm{Tr}_\alpha((T^* T)^{\frac{1}{2}} (T^* T)^{-\frac{1}{2}}) \quad \text{Cauchy-Schwarz}$$

$$= \mathrm{Tr}_\alpha(1_\alpha) = d(\alpha).$$

iff $(T^* T)^{\frac{1}{2}} = \lambda (T^* T)^{-\frac{1}{2}}$ for some $(\lambda \geq 0)$

$\rightarrow T^* T$ is a scalar

i.e. T = unitary $\times$ scalar. $\boxed{\text{}}$

$\alpha \in \mathcal{C}$, $\pi \in \mathrm{Irr}\,\mathcal{C}$ とする.

$\mathrm{Mor}(\pi, \alpha)$ は 内積

$$\langle S, T \rangle 1_\pi := \underset{\pi \leftarrow \alpha \leftarrow \pi}{T^* S} \in \mathrm{End}(\pi) = \mathbb{C}1_\pi$$

により fin. dim Hilb. sp.

ONB : V_n をとっておき

$$P_\pi^\alpha := \sum_n V_n V_n^* \in \mathrm{End}(\alpha)$$

projection をつくる. (ONBの取り方によらない) これは α の "isotypic component" への projection である.

Lem 1.21 P_π^α は α に関して natural

Proof.

$T : \alpha \to \beta$ をとる.

$$T P_\pi^\alpha = P_\pi^\beta T \quad \text{— (*)}$$ を確かめる.

$\mathrm{Mor}(\pi, \alpha)$ ONB V_n

$\mathrm{Mor}(\pi, \beta)$ ONB W_m

~~(*) $\Leftarrow$ W_m^*~~

$$T P_\pi^\alpha = \sum_n \underset{\pi \to \alpha \to \beta}{\boxed{T V_n}} V_n^*$$

$$= \sum_{m,n} \langle \cancel{W_m^*} T V_n, W_m \rangle W_m V_n^* \quad \text{スカラー}$$

$$\langle T V_n, W_m \rangle = (W_m^* T V_n)$$

$$= \sum_{m,n} W_m W_m^* T V_n V_n^*$$

$$= P_\pi^\beta T P_\pi^\alpha$$

$$\overset{\text{similarly}}{=} P_\pi^\beta T$$

Lem. 1.22

$\forall \alpha, \beta \in \mathcal{C}$

$\forall S : \alpha \to \beta$, $\forall T : \beta \to \alpha$

$$\mathrm{Tr}_\alpha(TS) = \mathrm{Tr}_\beta(ST)$$

Proof.

$$\begin{tikzcd} \gamma := & \alpha \arrow[l, bend right, "u"'] \oplus & \beta \arrow[ll, bend left, "v"] \end{tikzcd} \qquad \begin{tikzcd} \bar\gamma := & \bar\alpha \arrow[l, bend right, "u'"'] \oplus & \bar\beta \arrow[ll, bend left, "v'"] \end{tikzcd}$$

$$R_\gamma := (u' \otimes u) \underset{\uparrow \text{std.}}{R_\alpha} + (v' \otimes v) \underset{\uparrow \text{std}}{R_\beta}$$

$$\bar R_\gamma := (u \otimes u') \bar R_\alpha + (v \otimes v') \bar R_\beta$$

$\rightsquigarrow (R_\gamma, \bar R_\gamma)$ std sol. Check!

$$\hat S := v S u^* : \gamma \to \gamma$$

$$\hat T := u T v^* : \gamma \to \gamma$$

$$\begin{array}{ccc}
\mathrm{Tr}_\gamma(\hat S \hat T) & = & \mathrm{Tr}_\gamma(\hat T \hat S) \\
\| & & \| \\
\mathrm{Tr}_\gamma(v S T v^*) & & \mathrm{Tr}_\gamma(u T S u^*) \\
\| & & \| \\
R_\gamma^*(1_{\bar\gamma} \otimes v S T v^*) R_\gamma & & R_\gamma^*(1_{\bar\gamma} \otimes u T S u^*) R_\gamma \\
\| & & \| \\
R_\beta^*(1_{\bar\beta} \otimes S T) R_\beta & & R_\alpha^*(1_{\bar\alpha} \otimes T S) R_\alpha \\
\| & & \| \\
\mathrm{Tr}_\beta(ST) & & \mathrm{Tr}_\alpha(TS).
\end{array}$$

$\boxminus$

Frobenius map is unitary!

Section 2 ~~Approximation properties~~ of $\mathcal{C}$

cp-multiplier on $\mathcal{C}$ & admissible repn of $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

§2.1 Multipliers

$\mathcal{C}$: strict C^*-tensor category rig.d.

Defn. 2.1

A multiplier on $\mathcal{C}$ means a family $\{\theta_{\alpha,\beta}\}_{\alpha,\beta\in\mathcal{C}}$ s.t.

$$\theta_{\alpha,\beta} : \mathrm{End}(\alpha\otimes\beta) \to \mathrm{End}(\alpha\otimes\beta)$$

linear maps satisfying the following:

(i) Naturality w.r.t. (α,β)

$\forall\ \alpha_1 \xrightarrow{u} \alpha_2 \xrightarrow{u'} \alpha_1$

$\forall\ \beta_1 \xrightarrow{v} \beta_2 \xrightarrow{v'} \beta_1$

$$\theta_{(\alpha_2,\beta_2)}((u\otimes v)T(u'\otimes v'))$$

$$=(u\otimes v)\,\theta_{(\alpha_1,\beta_1)}(T)\,(u'\otimes v')$$

$\forall\ T\in\mathrm{End}(\alpha_1\otimes\beta_1)$

(ii) $\theta_{\alpha_2\otimes\alpha_1,\beta_1\otimes\beta_2}(1_{\alpha_2}\otimes T\otimes 1_{\beta_2})$

$$= 1_{\alpha_2}\otimes\theta_{\alpha_1,\beta_1}(T)\otimes 1_{\beta_2}$$

$\forall\ T\in\mathrm{End}(\alpha_1\otimes\beta_1)$

☆ $\theta_{\alpha,\beta}(1_{\alpha\otimes\beta}) = \lambda 1_{\alpha\otimes\beta}$ does not depend on α,β.

$$\theta_{\alpha\otimes 1,1\otimes\beta}(1_\alpha\otimes 1_1\otimes 1_1\otimes 1_\beta) = 1_\alpha\otimes\theta_{1,1}(1_1\otimes 1_1)\otimes 1_\beta$$

$$\theta_{1,1}(1_1\otimes 1_1) = \lambda\, 1_1\otimes 1_1$$

よりみち

こういう $\theta_{\alpha,\beta}$ の出所.

$N \subset M$ subfactor

$\rightsquigarrow$

standard λ-lattice

$\mathcal{G}_{N\subset M}$ $\quad A_{n,m} := M_n' \cap M_m$
$\quad n \le m$

$\cdots \subset M_{-2} \subset \underset{\substack{\| \\ M_{-1}}}{N} \subset M_0 \subset M_1 \subset \cdots$

$\mathcal{G}_{N\subset M}$ は 群のようなもの (paragrp)

☆ Popa. $\mathcal{G}$ の公理化.

・$\forall \mathcal{G}$ $\exists N\subset M$ $\mathcal{G} = \mathcal{G}_{N\subset M}$

ところで. discrete grp には.

amenability, Haagerup, (T), weak amenability etc

といった approx. property がある.

(multipliers $\varphi : \Gamma \to \mathbb{C}$ の性質で defn される)

$\mathcal{G}$ にもそのような multiplier を defn したい.

・Popa の strategy

$\mathcal{G}$ given data $\rightsquigarrow$ $\mathcal{G} = \mathcal{G}_{N\subset M}$

model subfactor

(not unique.
a model is non-hyperfin)

$\rightsquigarrow$ $N \subset M$ から SE-inclusion をつくる.

$$\begin{array}{ccc} T & \subset & S \\ \because\!\!\shortparallel & & \because\!\!\shortparallel \\ M \bar{\otimes} M^{op} & & M \boxtimes_{e_N} M^{op} \end{array}$$

SE-inclusion は

$$\text{“} \; T \subset T \rtimes \mathcal{G} \; \text{”}$$

みたいなものと考えられる.

ここで discrete grp の analogy

$$R \subset R \rtimes \Gamma$$

Γ 上の multiplier $\varphi : \Gamma \to \mathbb{C}$

$$\rightsquigarrow m_\varphi : R \rtimes \Gamma \to R \rtimes \Gamma$$

$$a\lambda(s) \longmapsto \varphi(s) a \lambda(s)$$

を参考にすると

$\theta : S \to S$ で T を各元不変にするもの

つまり

$$\theta(axb) = a\theta(x)b \quad {}^\forall a, b \in T, \; {}^\forall x \in S$$

という bimodular property をもつものが

$\mathcal{G}$ の multiplier と思える.

この θ を category の言葉でとらえると

先程の $\theta_{\alpha,\beta}$ たちができる.

$$\mathcal{C}_{N \subset M} \subset {}_M\mathrm{Hilb}_M$$

$\uparrow$

${}_M L^2 M_1{}_M$ が生成する ($\otimes_M$ して subobject をとっていく) C^*-tensor category とする

たとえば

$$\alpha := {}_M L^2 M_1 {}_M$$

として

$$\alpha \otimes \alpha = {}_M L^2 M_1 \underset{M}{\otimes} L^2 M_1 {}_M$$

$$\cong {}_M L^2 M_2 {}_M$$

であり，End をとると

$$\begin{array}{ccc} \mathrm{End}(\alpha \otimes \alpha) & \cong & \mathrm{End}({}_M L^2 M_2 {}_M) \\ & & \wr\| \\ & & M' \cap M_4 \\ & & \wr\| \\ & & M_{-2}' \cap M_2 \\ & & \cap \\ & & M \underset{e_N}{\boxtimes} M^{op} \circlearrowleft \Theta \end{array}$$

このように Θ は $\Theta_{\alpha,\alpha} \curvearrowright \mathrm{End}(\alpha \otimes \alpha)$

を誘導する．しかも．

$$\begin{array}{ccc} \mathrm{End}(\alpha \otimes \alpha) & \cong & M_{-2}' \cap M_2 \\ \cup & & \\ \mathrm{End}(\alpha) \otimes \mathrm{End}(\alpha) & \cong & (M_{-2}' \cap M) \vee (M' \cap M_2) \subset M^{op} \\ & & \cap \\ & & M \bar{\otimes} M^{op} \\ & & \| \\ & & T \end{array}$$

となることが分かり，$\Theta_{\alpha,\alpha}$ は $\mathrm{End}(\alpha) \otimes \mathrm{End}(\alpha)$-

bimodular であることがわかる．

このようにして

$$T \subset S \quad \text{の multiplier}$$

$$\rightsquigarrow \mathcal{C}_{N \subset M} \quad \text{の} \quad \text{—〃—}$$

ができる．

よりみち終わり

さて. $\mathcal{C}$ 上の multiplier と $\mathrm{Irr}\,\mathcal{C}$ 上の functor との 対応 が あることを みよう.

Lem. 2.2

$\theta = \{\theta_{\alpha,\beta}\}_{\alpha,\beta\in\mathcal{C}}$ multiplier on $\mathcal{C}$

Then $\forall \alpha, \beta, \gamma, \delta \in \mathcal{C}$

$\exists !$

$$\theta_{(\alpha,\beta),(\gamma,\delta)} : \mathrm{Mor}(\alpha\otimes\beta, \gamma\otimes\delta) \longrightarrow \mathrm{Mor}(\alpha\otimes\beta, \gamma\otimes\delta)$$

arrow　　arrow

linear map

s.t. (Naturality w.r.t. $\alpha, \beta, \gamma, \delta$)

$\forall\ u : \alpha_1 \to \alpha_2 \qquad s : \gamma_2 \to \gamma_1$

$\quad\ v : \beta_1 \to \beta_2 \qquad t : \delta_2 \to \delta_1$

$\forall\ X \in \mathrm{Mor}(\alpha_1\otimes\beta_1, \gamma_1\otimes\delta_1)$

$$\boxed{\begin{aligned}&\theta_{(\alpha_2,\beta_2),(\gamma_2,\delta_2)}((u\otimes v)X(s\otimes t))\\ &= (u\otimes v)\,\theta_{(\alpha_1,\beta_1),(\gamma_1,\delta_1)}(X)\,(s\otimes t)\end{aligned}}$$

Proof.

Let $\alpha, \beta, \gamma, \delta \in \mathcal{C}$.

$$\mu := \alpha \oplus \gamma \quad (u_\alpha,\ u_\gamma)$$
$$\nu := \beta \oplus \delta \quad (u_\beta,\ u_\delta)$$

isometries $\begin{cases} u_\alpha u_\alpha^* + u_\gamma u_\gamma^* = 1_\mu \\ u_\beta u_\beta^* + u_\delta u_\delta^* = 1_\nu \end{cases}$

$$\theta_{(\alpha,\beta),(\gamma,\delta)}(X) := (u_\alpha^*\otimes u_\beta^*)\,\theta_{\mu,\nu}((u_\alpha\otimes u_\beta)X(u_\gamma^*\otimes u_\delta^*))(u_\gamma\otimes u_\delta)$$

($\leadsto$ uniqueness).

$T : \alpha \to \alpha'$

$(T\otimes 1)\,\theta_{(\alpha,\beta),(\gamma,\delta)}(X)$

bimodularity

$= u_{\alpha'}^*(u_{\alpha'} T u_\alpha^* \otimes u_\beta)\,\theta_{\mu,\nu}(\ \text{—}\,\text{〃}\,\text{—}\ )\ \text{—}\,\text{〃}\,\text{—}$

$\blacksquare$

Now let $\theta \curvearrowright \mathcal{C}$ multiplier.

$\rightsquigarrow \theta \curvearrowright \mathrm{Mor}(\alpha\circ\beta,\ \gamma\circ\delta)$ た3
extends to $(\alpha,\beta),(\gamma,\delta)$ arrow

(Note: $\theta_{\alpha,\beta} = \theta_{(\alpha,\beta),(\alpha,\beta)}$).
original

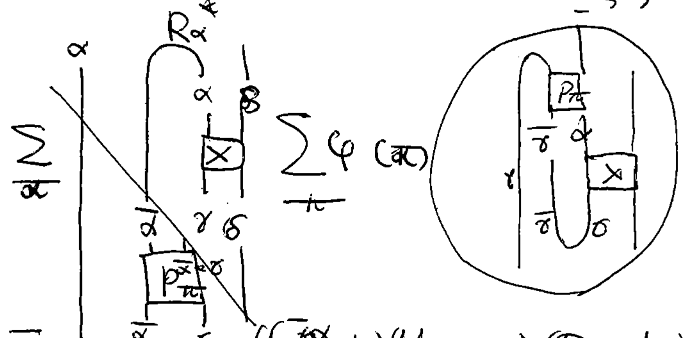
$\rightsquigarrow \theta_{(\alpha,\bar\alpha),(\mathbb{1},\mathbb{1})} \curvearrowright \mathrm{Mor}(\alpha\circ\bar\alpha,\ \mathbb{1}\circ\mathbb{1}) = \mathbb{C}\bar{R}_\alpha$ arrow one-dim.

$\exists\ \varphi^\theta(\alpha) \in \mathbb{C}$ s.t.

$$\theta_{(\alpha,\bar\alpha),(\mathbb{1},\mathbb{1})}(\bar{R}_\alpha) = \varphi(\alpha)\,\bar{R}_\alpha$$

$\varphi^\theta : \mathrm{Irr}\,\mathcal{C} \longrightarrow \mathbb{C}$.

この φ が θ を決めてしまうことを示す。

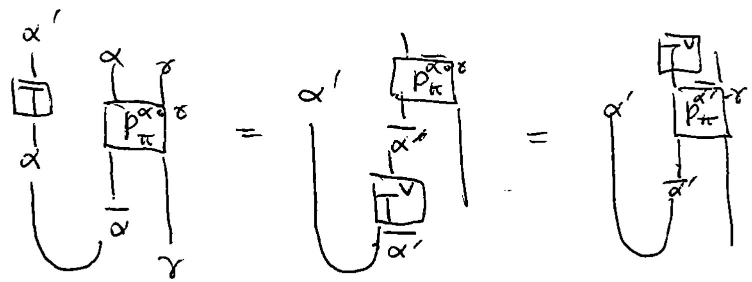
$\theta^\varphi_{(\alpha,\beta),(\gamma,\delta)}(X) :=$ 

$\bar{R}_\alpha$ $\sum_{\alpha}$ $\sum_{\pi} \varphi(\pi)$ $(\bar{R}^*_\gamma \cdot 1)(p_\pi^{\bar\alpha\alpha} \cdot 1_\beta)(1_{\bar\gamma} \otimes X)(R_\gamma \otimes 1_\delta)$ こちら

と定めると 明らかに, β, δ にかんして natural かつ.

Lem 1.21 より α に関しても natural.

γ に関しては, $T: \alpha \to \alpha'$ に対して (α のnaturality.)



であることと $T^{\vee\vee} = T$ (std. sol.) であることから従う.

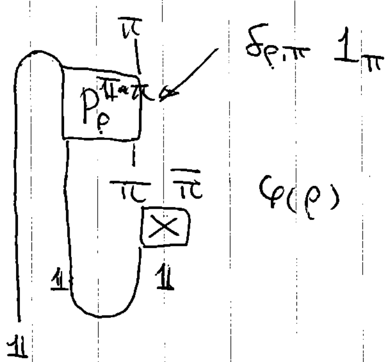
よって 対応

$$\theta \longrightarrow \varphi^\theta$$
$$\theta^\varphi \longleftarrow \varphi$$

が逆写像であることを示せばよい.

$\varphi \to \Theta^{\varphi} \to \psi$ と作ると.

$\pi \in \mathrm{Irr}\,\mathcal{C}$ に対し.

$\Theta^{\varphi}_{(\pi,\bar{\pi}),(1\!\!1,1\!\!1)}(X) = \sum_{\rho}$  $\varphi(\rho)$

$= \varphi(\pi)\, X$

である.

一方で $\theta \to \varphi \to \Theta^{\varphi}$ と作ると.

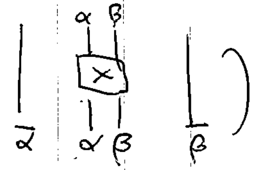
$$\Theta^{\varphi}_{\alpha,\beta} = \theta_{\alpha,\beta} \quad \forall\, \alpha, \beta \in \mathcal{C}$$

であることを示す.

$\Theta^{\varphi}_{\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta}}(1_{\bar{\alpha}} \otimes X \otimes 1_{\bar{\beta}})$

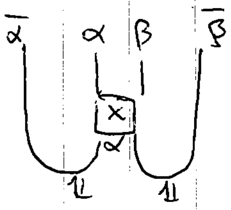
$\overset{?}{=} \theta_{\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta}}(1_{\bar{\alpha}} \otimes X \otimes 1_{\bar{\beta}})$ $\quad \forall X \in \mathrm{End}(\alpha\otimes\beta)$ より

diagram でかくと.

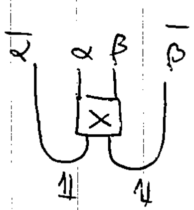
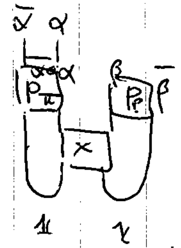
$\Theta^{\varphi}_{\bar{\alpha}\otimes\alpha,\beta\otimes\bar{\beta}}$ ()

$=$ $\Theta_{(\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta}),(\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta})}$

$\downarrow$ 右から $R_{\alpha} \otimes R_{\beta}$ をかけると naturality から

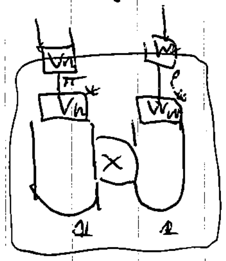
$\Theta^{\varphi}_{(\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta})(1\!\!1,1\!\!1)}$ ()

$\overset{?}{=} \theta^{\varphi}_{(\bar{\alpha}\circ\alpha,\beta\circ\bar{\beta}),(1\!\!1,1\!\!1)}$ (—//—) を示せばよい.

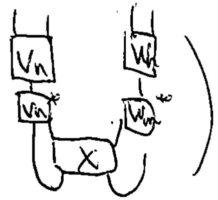
 $= \sum_{\pi,\rho}$ 

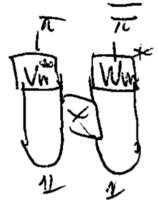
V_n ONB of $\mathrm{Mor}(\pi, \bar{\alpha}\circ\alpha)$

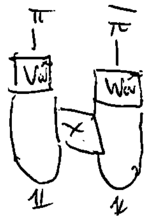
W_m ONB of $\mathrm{Mor}(\rho, \beta\otimes\bar{\beta})$

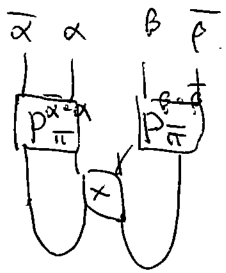
$= \sum_{\substack{\pi,\rho\\ n,m}}$  $\to \delta_{\rho,\bar{\pi}}$

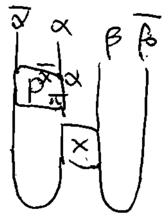
$\theta^{\varphi\,\text{トレ}}(\text{—〃—})$

$= \sum_{n,m,\pi} \theta^{\varphi\,\text{トレ}}_{(\bar{\alpha}\circ\alpha,\ \beta\circ\bar{\beta}),(1,1)}\Big($  $\Big)$

$= \sum_{n,m,\pi} (V_n \otimes W_n)\ \theta^{\varphi\,\text{トレ}}_{(\pi,\bar{\pi}),(1,1)}\Big($  $\Big)$

$=$ defn of $\varphi \rightarrow \varphi(\pi)$ 

$= \sum_{\pi} \varphi(\pi)$ 

$= \sum_{\pi} \varphi(\pi)$ 

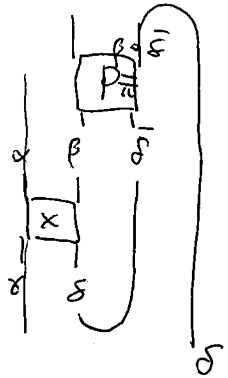
$= \theta^{\varphi}(\text{—〃—})$

まとめ.

$\varphi : \mathrm{Irr}\,\mathcal{C} \rightarrow \mathbb{C}$ function

$\overset{1:1}{\longleftrightarrow}$ $\theta^{\varphi}_{\alpha,\beta} \curvearrowright \mathrm{End}(\alpha\otimes\beta)$ multiplier

☆ Show

$\theta_{(\alpha,\beta)(\gamma,\delta)}(x) = \sum \varphi(\pi)$ 

So, any function $\varphi : \mathrm{Irr}\,\mathcal{C} \rightarrow \mathbb{C}$ is also called a multiplier on $\mathcal{C}$.

§ 2.2 cp-multipliers

$\mathcal{C}$ rigid strict C^*-tensor category

Defn. 2.3

$\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$

is said to be

(1) <u>cp-multiplier</u>

$\overset{\text{defn}}{\iff}$ $\Theta^{\varphi}_{\alpha,\beta} \curvearrowright \mathrm{End}(\alpha \otimes \beta)$ is c.p.
for $\forall\, \alpha, \beta \in \mathcal{C}$

(2) cb-multiplier

$\overset{\text{defn}}{\iff}$ $\Theta^{\varphi}_{\alpha,\beta} \curvearrowright \mathrm{End}(\alpha \otimes \beta)$ is c.b.
for $\forall\, \alpha, \beta \in \mathcal{C}$

☆ Recall

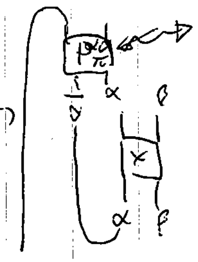
$$\Theta_{\alpha,\beta}(1_{\alpha\otimes\beta}) = \Theta_{\mathbb{1},\mathbb{1}}(1_{\mathbb{1}\otimes\mathbb{1}}) = \lambda$$

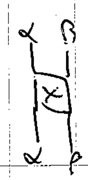
☆ $\varphi_n \xrightarrow[n]{} \varphi$ ptwise

$\iff$ $\Theta^{\varphi_n}_{\alpha,\beta} \xrightarrow[n]{} \Theta^{\varphi}_{\alpha,\beta}$ in norm $\forall\, \alpha, \beta \in \mathcal{C}$

<u>Ex. 2.4</u>

$\varphi_{\cancel{\sharp}}(\cancel{\pi}) := \cancel{\delta_{\pi,\mathbb{1}}}\ 1 \qquad \cancel{\pi} \in \mathrm{Irr}\,\mathcal{C}$

$\rightsquigarrow \Theta^{\varphi_{\cancel{\sharp}}}_{\alpha,\beta}(x) = \sum_{\pi} \varphi_{0}(\pi)$ 

$= $  $= x.$ $\quad \Theta^{\varphi_{\sharp}}_{\alpha,\beta} = \mathrm{id}$

<u>Ex 2.5</u> φ_0

Problem 2.6

どんな $\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ が c.p. か?

Prop 2.7

$\theta = \{\theta_{\alpha,\beta}\}_{\alpha,\beta\in\mathcal{C}}$ multiplier on $\mathcal{C}$.

T.F.A.E.

(1) $\theta_{\alpha,\beta}$ is c.p. for $\forall \alpha,\beta\in\mathcal{C}$

(2) $\theta_{\alpha,\beta}$ is positive — " —

(3) $\theta_{\alpha,\bar{\alpha}}(\bar{R}_\alpha \bar{R}_\alpha^*) \geq 0$ for $\forall \alpha\in\mathcal{C}$

Proof

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial. (2) $\Rightarrow$ (1)

$$\begin{array}{cc} M_n(\mathbb{C})\otimes \mathrm{End}(\alpha\otimes\beta) & id\otimes\theta_{\alpha,\beta} \\ \cong & \parallel \\ \mathrm{End}(\alpha^{\oplus n}\otimes\beta) & \theta_{\alpha^{\oplus n},\beta} \end{array}$$

(3) $\Rightarrow$ (2)

Let $T\in \mathrm{End}(\alpha\otimes\beta)$.

Claim. $\exists \gamma\in\mathcal{C}$ s.t.
$\exists S\in \mathrm{Mor}(\alpha,\alpha\otimes\gamma)\otimes \mathrm{Mor}(\beta,\bar{\gamma}\otimes\beta)$

s.t.

$$T = (1_\alpha\otimes \bar{R}_\gamma^* \otimes 1_\beta)\cdot S$$

If this is ok, then

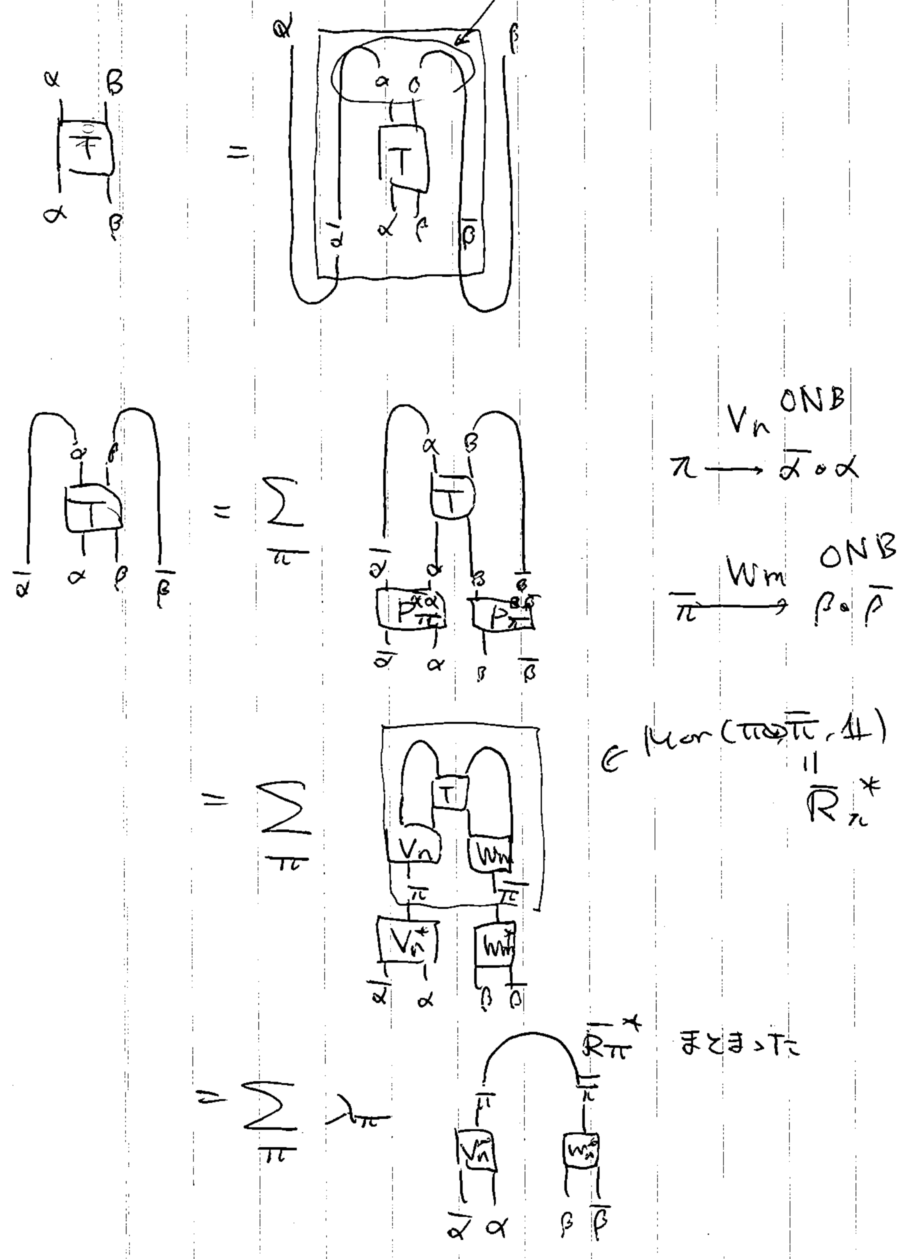
$$\theta_{\alpha,\beta}(T^*T) = \theta_{\alpha,\beta}(S^*(1_\alpha\otimes \bar{R}_\gamma \bar{R}_\gamma^* \otimes 1_\beta)S)$$

$$= S^*\theta_{\alpha\otimes\gamma,\bar{\gamma}\otimes\beta}(1_\alpha\otimes \bar{R}_\gamma \bar{R}_\gamma^* \otimes 1_\beta)S$$

$$= S^*(1_\alpha\otimes \theta_{\gamma,\bar{\gamma}}(\bar{R}_\gamma \bar{R}_\gamma^*)\otimes 1_\beta)S$$

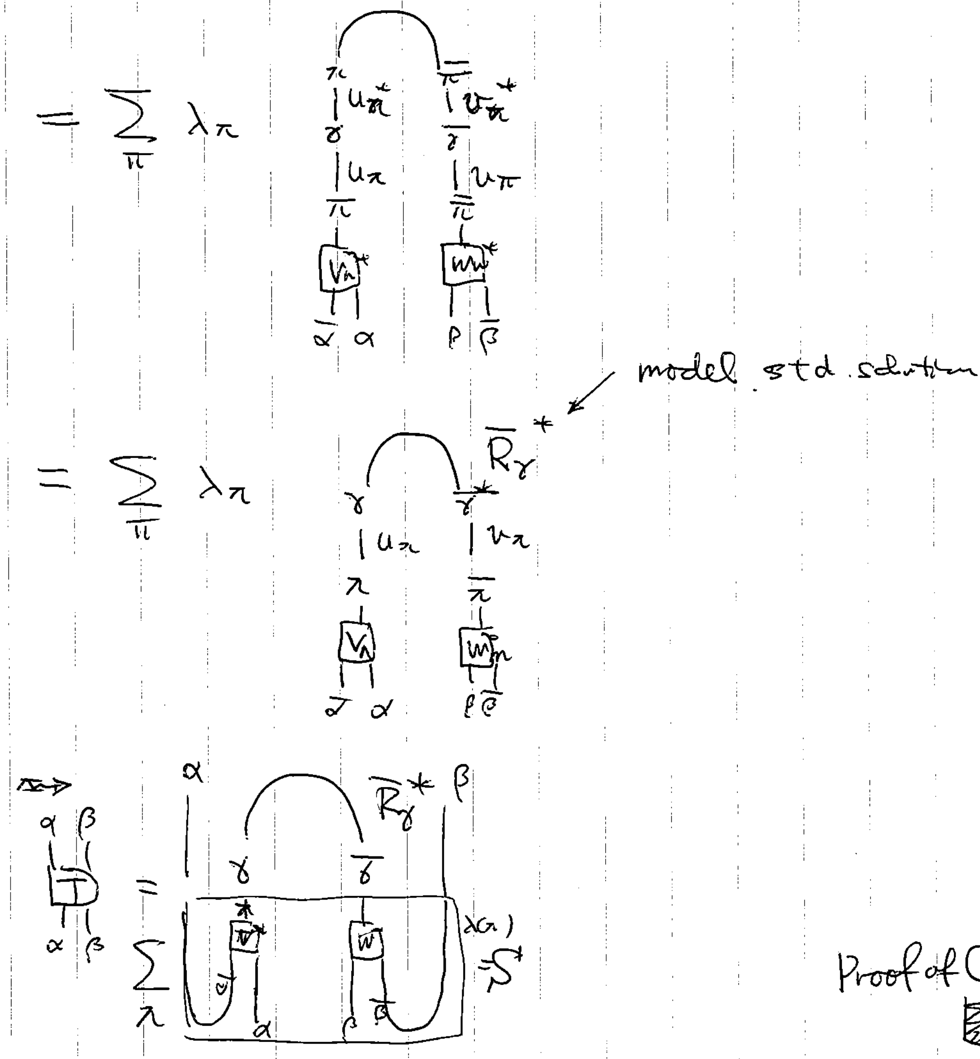
$$\geq 0.$$

Proof of Claim



$\gamma := \bigoplus_{\substack{\pi\in\mathrm{Irr}\,\mathcal{C} \\ \pi\prec\bar{\alpha}\circ\alpha}} \pi$

isometry $\pi \xrightarrow{u_\pi} \gamma$, $\bar{\pi} \xrightarrow{v_\pi} \bar{\gamma}$

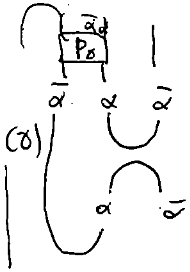


Proof of Claim ∎

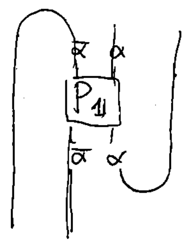
Proof of Prop. ∎

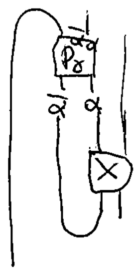
<u>Ex.</u> 2.5

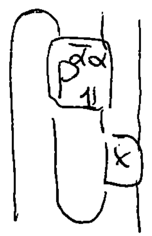
$\varphi_0(\alpha) := \delta_{\alpha,1\!\!1}$ $\quad \alpha \in \mathrm{Irr}\,\mathcal{C}$. is c.p.

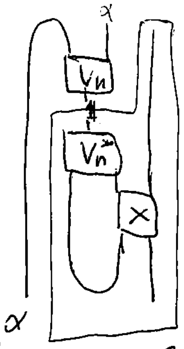
$\Theta^{\varphi_0}_{\alpha,\bar\alpha}(\bar R_\alpha \bar R_\alpha^*) = \sum_\gamma \varphi_0(\gamma)$ 

$\alpha \in \mathcal{C}$

$=$ 

$\Theta^{\varphi_0}_{\alpha,\beta}(x) = \sum_\gamma$  $\varphi_0(\gamma)$

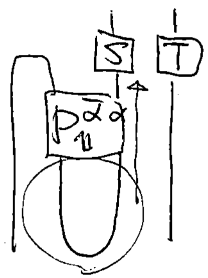
$=$ 

$= \sum_n$  $\quad V_n$ ONB $1\!\!1 \to \bar\alpha \circ \alpha$

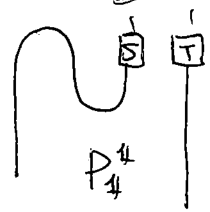
End(α) $\quad \in$ End(β)

$\in \mathrm{End}(\alpha) \otimes \mathrm{End}(\beta)$

if $X = S \circ T$

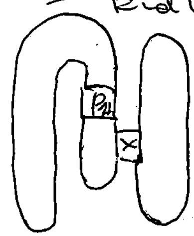
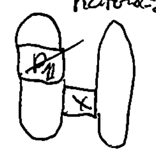
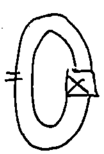
$\Theta^{\varphi_0}_{\alpha,\beta}(x) =$  (naturality)

Lem 1.21

$=$  (naturality)

$= S \circ T$.

$\to \Theta^{\varphi_0}_{\alpha,\beta} : \mathrm{End}(\alpha \circ \beta) \longrightarrow \mathrm{End}(\alpha) \circ \mathrm{End}(\beta)$ $\quad$ cond. exp. trace preserving

$(\mathrm{Tr}_\alpha \circ \mathrm{Tr}_\beta)(\Theta^{\varphi_0}_{\alpha,\beta}(x)) =$  $\overset{\text{naturality}}{=}$  $=$  $= \mathrm{Tr}_{\alpha\beta}(x)$.

§.2.3 Fusion $*$-algebra & its repn

Γ : discrete grp

$\varphi : \Gamma \to \mathbb{C}$ positive definite

$\Longleftrightarrow$ $\exists\, \pi : \Gamma \longrightarrow B(H_\pi)$ unitary repn

$\exists\, \xi \in H_\pi$

s.t.

$$\varphi(s) = \langle \pi(s)\xi, \xi \rangle \quad \forall s \in \Gamma.$$

これと同じようなことを $\mathcal{C}$ でもやりたい!

group algebra $\mathbb{C}\Gamma$

$\quad\wr$ 一般化

vector space. $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] = \mathrm{span}\{ \cancel{[\alpha]}\ \alpha \mid \alpha \in \mathrm{Irr}\,\mathcal{C} \}$

- product : $\underset{\in \mathrm{Irr}\,\mathcal{C}}{\alpha \cdot \beta} := \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\alpha,\beta}^{\gamma}\, \gamma$,

where $N_{\alpha,\beta}^{\gamma}$ = the multiplicity of γ in $\alpha \otimes \beta$

$= \dim \mathrm{Mor}(\gamma, \alpha \otimes \beta)$.

- $*$: $\alpha^* := \overline{\alpha}$

conj. object of α

Defn. 2.8

$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$ を fusion $*$-algebra とよぶ.

☆ $\mathbb{1}$ is unit $\quad \mathbb{1}\cdot\alpha = \alpha$

Defn. 2.9

> A $*$-homo. $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(H)$ is called a <u>repn</u>.

cp-multiplier φ on $\mathcal{C}$ $\overset{\text{OK}}{\rightsquigarrow}$ repn. of $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

$\not\leftsquigarrow$ Not in general.

Hom の構造を忘れている!

Later $\forall\varphi$ cp-multiplier

$\exists\,\pi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(H)$ repn

$\exists\,\xi \in H$

s.t. $\quad \varphi(\alpha) = \dfrac{1}{d(\alpha)}\langle \pi(\alpha)\xi, \xi\rangle \quad \forall\alpha\in\mathrm{Irr}\,\mathcal{C}$.

Ex. 2.10

$\mathcal{C} = \mathrm{Rep}\,SU(2)$

$\mathrm{Irr}\,\mathcal{C} = \{\pi_0, \pi_{1/2}, \pi_1, \cdots\}$

trivial repn $\mathbb{1}$; fundamental repn ; half integers

$\dim H_{\pi_\nu} = 2\nu+1$

$\pi_\mu\cdot\pi_\nu = \pi_{\mu+\nu} + \pi_{\mu+\nu-1} + \cdots + \pi_{|\mu-\nu|}$

$\pi_{1/2}\cdot\pi_{1/2} = \pi_1 + \pi_0 \quad \rightarrow \overline{\pi_{1/2}} = \pi_{1/2}$

$\pi_\mu\cdot\pi_{1/2} = \pi_{\mu+1/2} + \pi_{\mu-1/2}$

$\rightsquigarrow \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] = \mathbb{C}[\pi_{1/2}]$ $\quad$ $\pi_{1/2}$ で生成.

$\cong \mathbb{C}[x]$ $\quad$ polynomial alg $\quad x^* = x$.

☆ No "universal C^*-alg" of $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

like $\quad \|x\| := \sup_{\text{all repn }\pi} \|\pi(x)\|$

Ex. 2.11 (Dimension function).

$\rho : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ is <u>dim. funct.</u>

$\overset{\text{defn}}{\Longleftrightarrow}$
$$\begin{cases} \cdot\ \rho(\alpha)\rho(\beta) = \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\alpha\beta}^{\gamma}\, \rho(\gamma) & \forall \alpha, \beta \in \mathrm{Irr}\,\mathcal{C}. \\ \cdot\ \rho(\bar{\alpha}) = \rho(\alpha) \\ \cdot\ \rho(\alpha) \geqq 0 \end{cases}$$

induces $\rightsquigarrow$ $\rho : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to \mathbb{C}$ repn.

☆ $\rho(\mathbb{1}) = 1$

☆ $\alpha\bar{\alpha} = \mathbb{1} + \cdots$

$$\rho(\alpha)\rho(\bar{\alpha}) = 1 + \underbrace{\cdots}_{+}$$
$$\geqq 1$$

$\rightsquigarrow$ $\rho(\alpha) \geqq 1 \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}.$

☆ The intrinsic dim is a dim funct.

Ex. 2.12 (Regular repn.)

$$\ell^2(\mathrm{Irr}\,\mathcal{C})$$
$$\cup \text{ dense}$$
$\pi_{\mathrm{reg}}(\alpha) \curvearrowright \mathrm{span}\{\delta_\beta \mid \beta \in \mathrm{Irr}\,\mathcal{C}\}$

$$\pi_{\mathrm{reg}}(\alpha)\cdot\delta_\beta := \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\alpha\beta}^{\gamma}\,\delta_\gamma \qquad \alpha, \beta \in \mathrm{Irr}\,\mathcal{C}.$$

☆ $\pi_{\mathrm{reg}}(\alpha)$ is bounded &

$\|\pi_{\mathrm{reg}}(\alpha)\| \leqq \rho(\alpha)$ for $\forall \alpha \in \mathrm{Irr}\,\mathcal{C}$
$\forall \rho$ dim funct.

Lem. 2.~~10~~13

$A := (a_{ij})_{i,j \in I}$ (index set) with $a_{ij} \geqq 0$.

If $\exists\, u = (u_i)_{i \in I}$ s.t.

- $u_i > 0, \ \forall i \in I$
- $v := Au$ well-defn. (i.e. $\sum_j A_{ij} u_j < \infty \ \forall i$)
- ${}^{t}A v \leqq u$ coordinate-wise

Then A defines a contraction on $\ell^2(I)$

Let $A := (A_{\beta,\gamma})_{\beta,\gamma \in \mathrm{Irre}}$

$$A_{\beta,\gamma} := \frac{N_{\alpha\gamma}^{\beta}}{\rho(\alpha)}$$

$$u := (\rho(\beta))_{\beta}$$

$$=: v$$

$$[Au]_{\beta} = \sum_{\gamma} A_{\beta\gamma} u_{\gamma}$$

$$= \sum_{\gamma} \frac{N_{\alpha\gamma}^{\beta}}{\rho(\alpha)} u_{\gamma} \qquad N_{\bar{\alpha}\beta}^{\gamma} \qquad u_\gamma = \rho(\gamma)$$

$$= \frac{\rho(\cancel{\bar{\alpha}})\rho(\beta)}{\rho(\cancel{\alpha})}$$

$$= v_{\beta}$$

$$[{}^tA v]_{\beta} = \sum_{\gamma} [{}^tA]_{\beta\gamma} v_{\gamma}$$

$$= \sum_{\gamma} A_{\gamma\beta} v_{\gamma}$$

$$= \sum_{\gamma} \frac{N_{\alpha\beta}^{\gamma}}{\rho(\alpha)} \rho(\gamma)$$

$$= \frac{\rho(\cancel{\alpha})\rho(\beta)}{\rho(\cancel{\alpha})}$$

$$= u_{\beta}$$

By Lem, $\|A\| \leqq 1$.

$$\frac{\pi_{\mathrm{reg}}(\alpha)}{\rho(\alpha)} \delta_{\gamma} = \frac{1}{\rho(\alpha)} \sum_{\beta} N_{\alpha\gamma}^{\beta} \delta_{\beta}$$

$$= \sum_{\beta} A_{\beta\gamma} \delta_{\beta}$$

$$= A \delta_{\gamma}$$

$$\rightsquigarrow \|\pi_{\mathrm{reg}}(\alpha)\| \leqq \rho(\alpha).$$

Proof of Lem 2.13

~~Let~~ Note if $v_{i_0} = 0$ for some i_0 then $A_{ij} = 0$ $\forall j \in I$

$\to$ We may replace $v_{i_0} = 1$. $\xrightarrow{\text{WMA}}$ $v_i > 0$ $\forall i \in I$

Let $\xi, \eta \in \operatorname{span}\{\delta_k \mid k \in I\}$.

$$|\langle A\xi, \eta\rangle| = \left|\sum_{i,j} A_{ij} \xi_j \overline{\eta_i}\right|$$

$$= \left|\sum_{i,j} (A_{ij} v_i u_j^{-1})^{\frac{1}{2}} \xi_j \cdot \overline{(A_{ij} v_i^{-1} u_j)^{\frac{1}{2}} \eta_i}\right|$$

$$\leq \left(\sum_{i,j} A_{ij} v_i u_j^{-1} |\xi_j|^2\right)^{1/2} \cdot \left(\sum_{i,j} A_{ij} v_i^{-1} u_j |\eta_i|^2\right)^{1/2}$$

$$= \left(\sum_j [{}^tAv]_j u_j^{-1} |\xi_j|^2\right)^{1/2} \cdot \left(\sum_{i,j} [Au]_i v_i^{-1} |\eta_i|^2\right)^{1/2}$$

$$\leq \|\xi\| \, \|\eta\| \qquad \blacksquare$$

Now we introduce the admissibility of repns of $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

Defn. 2.14 (Admissibility)

> A repn. $\pi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(H)$ is <u>admissible</u> w.r.t. $\mathcal{C}$
>
> if the functions
>
> $$\varphi(\alpha) := \frac{1}{d(\alpha)} \langle \pi(\alpha)\xi, \xi\rangle, \quad \alpha \in \mathrm{Irr}\,\mathcal{C}$$
>
> are c.p.-multipliers for $\forall \xi \in H$

☆ $\mathrm{Rep}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]) \cup \mathrm{Rep}_{\mathcal{C}\text{-adm}}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])$ full C^*-subcategory

Ex. 2.15

intrinsic dim

$d : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow \mathbb{C} = B(\mathbb{C})$

$\in \mathrm{Rep}_{\mathrm{adm}}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])$

$$\varphi(\alpha) = \frac{1}{d(\alpha)} d(\alpha) = 1 \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}$$

$$\rightsquigarrow \theta^{\varphi}_{\alpha,\beta} = \mathrm{id} \quad \forall \alpha, \beta \in \mathcal{C}.$$

Ex. 2.16

$\pi_{\mathrm{reg}} : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to B(\ell^2(\mathrm{Irr}\,\mathcal{C}))$

is admissible

$$\frac{1}{d(\alpha)} \langle \pi_{\mathrm{reg}}(\alpha) \delta_{\mathbb{1}}, \delta_{\mathbb{1}} \rangle = \frac{\delta_{\mathbb{1},\alpha}}{d(\alpha)} = \delta_{\alpha,\mathbb{1}} \cdot f^{\alpha}$$

$$= \varphi_0(\alpha). \quad \alpha \in \mathrm{Irr}\,\mathcal{C}$$

cyclic

See the ~~following~~ results 2.17 Lem

さて、~~$\varphi \in \mathrm{Irr}$~~

$\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$

$\overset{1:1}{\leftrightsquigarrow}$ $\omega_\varphi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to \mathbb{C}$ functional

$$\omega_\varphi(\alpha) := d(\alpha)\varphi(\alpha) \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}$$

というものを考える。このとき

$\varphi : \mathrm{Irr}\,\mathcal{C} \longrightarrow \mathbb{C}$ cp-mult.

$\Rightarrow \omega_\varphi \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]^*$ is positive

$$\omega_\varphi(x^* x) \geq 0 \quad \forall x \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}].$$

であることを示す。

$\rightsquigarrow$ GNS repn により $\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to B(H_{\omega_\varphi})$

ができる。

Lem. 2.17

$\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ given

$\omega_\varphi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to \mathbb{C}$ associated functional

For every $x, y \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$, define

$\varphi_{x,y} : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ s.t.

$$\omega_{\varphi_{x,y}}(a) = \omega_\varphi(y^* a x) \quad \text{for } \forall a \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$$

(1) $$\varphi_{x,y}(\rho) = \frac{1}{d(\rho)} \sum_{\pi,\eta,\gamma \in \mathrm{Irr}\,\mathcal{C}} x_\pi \overline{y_\eta}\, d(\gamma)\, \varphi(\gamma)\, \mathrm{mult}(\gamma, \overline{\eta} \otimes \rho \otimes \pi). \quad \forall \rho \in \mathrm{Irr}\,\mathcal{C}$$

(2) $$\Theta^{\varphi_{x,y}}_{\alpha,\beta}(X) = \sum_{\pi,\eta \in \mathrm{Irr}\,\mathcal{C}} x_\pi \overline{y_\eta}\, (1_\alpha \otimes \overline{R}_\pi^* \otimes 1_\beta) \cdot \Theta^{\varphi}_{(\alpha\otimes\pi),(\overline{\pi}\otimes\beta)),(\alpha\otimes\eta,\overline{\eta}\otimes\beta)}\big((1_\alpha \otimes \overline{R}_\pi \otimes 1_\beta) X (1_\alpha \otimes \overline{R}_\eta^* \otimes 1_\beta)\big) \cdot (1_\alpha \otimes \overline{R}_\eta \otimes 1_\beta)$$

Proof.

(1) $x = \sum x_\pi \pi$

$y = \sum y_\eta \eta$

$$y^* \rho x = \sum x_\pi \overline{y_\eta}\ \overline{\eta} \rho \pi = \sum x_\pi \overline{y_\eta}\, \mathrm{mult}(\gamma, \overline{\eta} \otimes \rho \otimes \pi)\, \gamma$$

$$\varphi_{x,y}(\rho) = \frac{1}{d(\rho)} \omega_{\varphi_{x,y}}(\rho) = \frac{1}{d(\rho)} \omega_\varphi(y^* \rho x) = \frac{1}{d(\rho)} \sum_\gamma x_\pi \overline{y_\eta}\, \mathrm{mult}(\gamma, \overline{\eta} \otimes \rho \otimes \pi)\, d(\gamma)\, \varphi(\gamma)$$

$\lrcorner$

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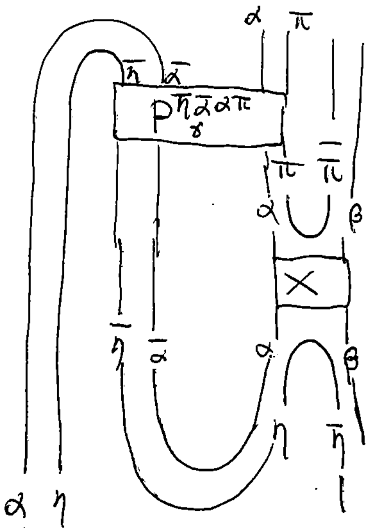
(2)

$$\Theta^{\varphi}_{(\alpha\circ\pi,\,\bar\pi\circ\beta),\,(\alpha\circ\eta,\,\bar\eta\circ\beta)}\Big(\big(1_\alpha\circ\bar R_\pi^{*}\circ 1_\beta\big)\times\big(1_\alpha\circ\bar R_\eta^{*}\circ 1_\beta\big)\Big)$$

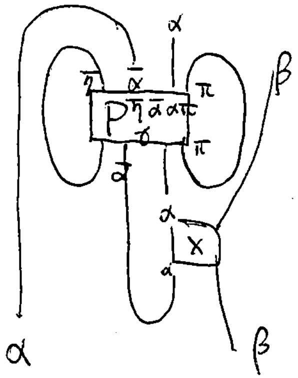
$= \sum_{\gamma}$



$\varphi(\gamma)$

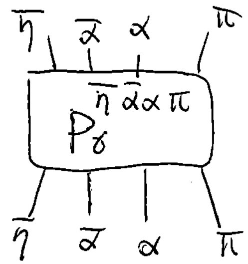
$$(1_\alpha\circ\bar R_\pi^{*}\circ 1_\beta)\,\Theta^{\varphi}(\;—〃—\;)\,(1_\alpha\circ\bar R_\eta\otimes 1_\beta)$$

$= \sum_{\gamma}\varphi(\gamma)$



— ①

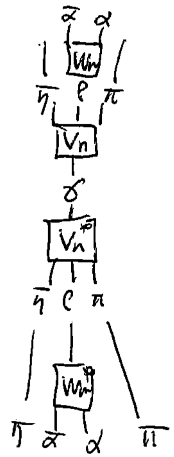
ここで、



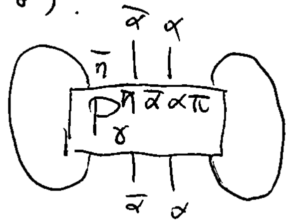
$$\begin{tikzcd} \gamma \arrow[r] \arrow[d, "V_n"', "\mathrm{ONB}"] & \bar\eta\bar\alpha\alpha\pi \\ \bar\eta\rho\pi \arrow[ur] & \end{tikzcd}$$

$$\rho \xrightarrow[\;W_m\;]{\mathrm{ONB}} \bar\alpha\alpha$$

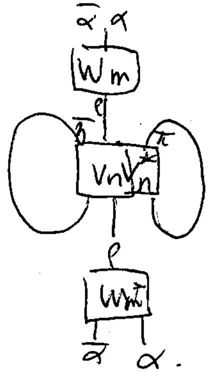
$= \sum_{\rho,n,m}$



より、

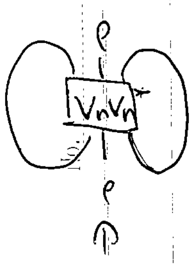


$= \sum_{\rho,n,m}$



— ②

$\exists\, 3\ \text{th}^m$

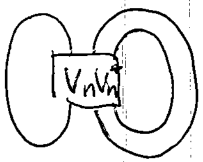


$= \lambda_\gamma 1_\rho \quad \exists \in \mathbb{C}$

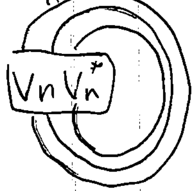
$\mathrm{End}(\rho) = \mathbb{C}1_\rho$

$\xrightarrow{\mathrm{Tr}_\rho}$

$= \lambda_\gamma d(\rho)$



$\|$

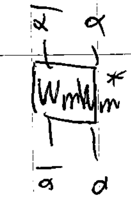


$= \mathrm{Tr}_{\bar\eta * \rho \circ \bar\pi}(V_n V_n^*)$

$= \mathrm{Tr}_\gamma(V_n^* V_n)\ 1_\gamma$

$= d(\gamma)$

$\rightsquigarrow \lambda_\gamma = \dfrac{d(\gamma)}{d(\rho)}$

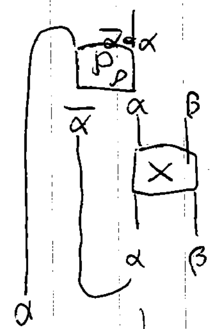
$\rightsquigarrow$ ② $= \displaystyle\sum_{\rho, n, m} \frac{d(\gamma)}{d(\rho)}$ 

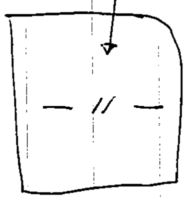
$= \displaystyle\sum_{\rho, m} \frac{d(\gamma)}{d(\rho)} \mathrm{mult}(\gamma, \bar\eta * \rho \circ \bar\pi)$ 

$= \displaystyle\sum_{\rho} \frac{d(\gamma)}{d(\rho)} \mathrm{mult}(\gamma, \bar\eta * \rho \circ \bar\pi)\ p_\rho^{\bar\alpha \circ \alpha}$

Hence

① $= \displaystyle\sum_{\gamma, \rho} \frac{d(\gamma)\varphi(\gamma)}{d(\rho)} \mathrm{mult}(\gamma, \bar\eta \circ \rho \circ \bar\pi)$



(2) の RHS $\rightsquigarrow \cancel{\theta^{\varphi_{x,y}}_{\alpha,\beta}(X)} = \displaystyle\sum_\rho \varphi_{x,y}(\rho)$  $= \theta^{\varphi_{x,y}}_{\alpha,\beta}(X)$

☆ φ : cp-multiplier

$\Rightarrow \varphi_{x,x}$: cp-mult. for $\forall x \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

Indeed. let $x = \sum_\pi x_\pi \pi \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

$$\delta := \bigoplus_{\substack{\pi \\ \pi \in \mathrm{supp}(x)}} \pi$$

$$\pi \xrightarrow{u_\pi} \delta \qquad \bar{\pi} \xrightarrow{v_\pi} \bar{\delta}$$

$$\bar{R}_\delta := \sum_{\pi \in \mathrm{supp}(x)} (u_\pi \otimes v_\pi) \cdot \bar{R}_\pi$$

$$S := \sum x_\pi (u_\pi \otimes v_\pi) \cdot \bar{R}_\pi$$

$$(1_\alpha \otimes \bar{R}_\delta^* \otimes 1_\beta).$$

$\rightsquigarrow$

$$\Theta^{\varphi_{x,x}}_{\alpha,\beta}(X) = \Theta^{\varphi}_{\alpha\otimes\delta,\,\bar{\delta}\otimes\beta}\Big(\big(1_\alpha \otimes S \otimes 1_\beta\big) X \big(1_\alpha \otimes S^* \otimes 1_\beta\big)$$

$$\cdot\big(1_\alpha \otimes \bar{R}_\delta \otimes 1_\beta\big)\Big)$$

Now, let $\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ cp-mult.

$\rightsquigarrow \omega_\varphi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to \mathbb{C}$.

Then for $\forall x \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

$$\omega_\varphi(x^* \cdot x) = \omega_{\varphi_{x,x}}(\mathbb{1})$$

$$= \overset{\overset{1}{\shortparallel}}{d(\mathbb{1})}\, \varphi_{x,x}(\mathbb{1})$$

$$\Theta^{\varphi_{x,x}}_{\mathbb{1},\mathbb{1}}(1_{\mathbb{1}\otimes\mathbb{1}})$$

$$\geq 0$$

$$\langle \hat{x}, \hat{y} \rangle_{\omega_\varphi} := \omega_\varphi(y^* x) \qquad x, y \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$$

positive semi-definite

$$\rightsquigarrow H_{\omega_\varphi} = \overline{\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]}^{\|\cdot\|_{\omega_\varphi}}$$

Hilbert space

$\pi_{\omega_\varphi} : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to B(H_{\omega_\varphi})$?

$\pi_{\omega_\varphi}(x)\,\hat{y} := \widehat{xy}$ well-defined? $\forall y \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$.

$$\langle \widehat{xy}, \widehat{xy} \rangle_{\omega_\varphi} = \omega_\varphi(y^* x^* x y)$$

$$= \omega_{\varphi_{y,y}}(x^* x)$$

$$\langle \widehat{\alpha y}, \widehat{\alpha y} \rangle_{\omega_\varphi} = \omega_{\varphi_{y,y}}(\bar{\alpha} \cdot \alpha) \qquad \alpha \in \mathrm{Irr}\,\mathcal{C}$$

$$= \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\bar{\alpha},\alpha}^{\gamma}\, \omega_{\varphi_{y,y}}(\gamma)$$

$$= \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\bar{\alpha},\alpha}^{\gamma}\, d(\gamma)\, \varphi_{y,y}(\gamma)$$

$$\leq \sum_{\gamma \in \mathrm{Irr}\,\mathcal{C}} N_{\bar{\alpha},\alpha}^{\gamma}\, d(\gamma)\, \varphi_{y,y}(\mathbb{1})$$

Lem ~~Claim~~ by ~~(*)~~ in the following

$\varphi_{y,y}(\mathbb{1}) = \omega_\varphi(y^* y) = \langle \hat{y}, \hat{y} \rangle$

$$= d(\alpha)^2 \langle \hat{y}, \hat{y} \rangle$$

$\to$ $\pi_{\omega_\varphi}(\alpha)$ well-defined

& $\|\pi_{\omega_\varphi}(\alpha)\| \leq d(\alpha)$ $\forall \alpha \in \mathrm{Irr}\,\mathcal{C}$.

Lema 2.18

~~Claim~~. $\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$ cp-mult.

$$\boxed{\Rightarrow |\varphi(\alpha)| \leq \varphi(\mathbb{1}) \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}}$$

Proof.

$$\Theta_{(\alpha,\bar{\alpha}),(\mathbb{1},\mathbb{1})}(\bar{R}_\alpha) = \varphi(\alpha)\,\bar{R}_\alpha$$

$\beta := \alpha \oplus \mathbb{1} \xleftarrow{u} \alpha$, $\xleftarrow{u} \mathbb{1}$

$\bar{\beta} := \bar{\alpha} \oplus \mathbb{1} \xleftarrow{u^-} \bar{\alpha}$, $\xleftarrow{u^-} \mathbb{1}$.

$$\mathrm{LHS} = (u^* \otimes v^*)\, \theta_{\beta,\bar\beta}\big((u \otimes v)\bar R_\alpha (u'^* \otimes v'^*)\big)(u'^* \otimes v'^*)$$

$$\|\mathrm{LHS}\| \leq \theta_{\beta,\bar\beta}(1_{\beta \otimes \bar\beta}) = \varphi(\mathbb{1})$$

$$\|\mathrm{RHS}\| = |\varphi(\alpha)|$$

Proof of ~~Claim~~ Lem. $\blacksquare$

Lem 2.19 $\boxed{\pi_{\omega_\varphi} : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(H_{\omega_\varphi}) \text{ is admissible}}$

Proof.

Let $\xi \in H_{\omega_\varphi} \longleftarrow \hat{y}_n \qquad y_n \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$

$$\psi(\alpha) := \frac{1}{d(\alpha)} \langle \pi_{\omega_\varphi}(\alpha)\xi, \xi \rangle$$

$$\leftarrow \psi_n(\alpha) := \frac{1}{d(\alpha)} \langle \pi_{\omega_\varphi}(\alpha)\, \hat{y}_n, \hat{y}_n \rangle$$

pt wise for $\alpha \in \mathrm{Irr}\,\mathcal{C}$

$\rightsquigarrow \theta^{\psi_n}_{\beta,\gamma} \longrightarrow \theta^{\psi}_{\beta,\gamma} \quad \forall \beta, \gamma \in \mathcal{C}$

Suffices to show $\theta^{\psi_n}_{\beta,\gamma}$ c.p.

$$\psi_n(\alpha) = \frac{1}{d(\alpha)} \omega_\varphi(y_n^* \alpha y_n)$$

$$= \frac{1}{d(\alpha)} \omega_{\varphi_{y_n y_n}}(\alpha)$$

Hence ψ_n is c.p. mult. $\blacksquare$

$\psi_n = \varphi_{y_n, y_n}$

☆ $\forall \varphi : \mathrm{Irr}\,\mathcal{C} \longrightarrow \mathbb{C}$ c.p. mult

is of the form

$$\varphi(\alpha) = \frac{1}{d(\alpha)} \langle \pi(\alpha)\xi, \xi \rangle \qquad \alpha \in \mathrm{Irr}\,\mathcal{C}.$$

admissible repn.

Ex.

$\ast$ $\varphi_0(\alpha) = \delta_{\alpha,\mathbb{1}} \rightsquigarrow \langle \alpha, \beta \rangle_{\omega_{\varphi_0}} = \omega_{\varphi_0}(\bar\beta\alpha) = \sum N^{\gamma}_{\bar\beta\alpha}\, \underset{d(\gamma)\varphi_0(\gamma)}{\omega_{\varphi_0}(\gamma)}$
$= \delta_{\alpha,\beta} 1. \quad \rightsquigarrow H_{\omega_{\varphi_0}} = \ell^2(\mathrm{Irr}\,\mathcal{C})$

$\ast$ $\varphi(\alpha) \equiv 1 \rightsquigarrow \langle \alpha, \beta \rangle_{\omega_\varphi} = d(\alpha)\, d(\beta) \rightsquigarrow \hat\alpha = d(\alpha)\hat{\mathbb{1}}$

Problem 2.20 What is an admissible repn? w.r.t. $\mathcal{C}$

☆ Let $\pi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(H)$ adm. repn w.r.t. $\mathcal{C}$

Then $\forall \xi$, $\underset{\varphi_\xi}{\alpha \longmapsto} \frac{1}{d(\alpha)} \langle \pi(\alpha)\xi, \xi \rangle$ is c.p. ($\alpha \in \mathrm{Irr}\,\mathcal{C}$)

$$\|\pi(\alpha)\xi\|^2 = \langle \pi(\bar{\alpha}\alpha)\xi, \xi\rangle$$

$$= \sum N_{\bar{\alpha}\alpha}^{\gamma} \langle \pi(\gamma)\xi, \xi\rangle$$

$$= \sum N_{\bar{\alpha}\alpha}^{\gamma}\, d(\gamma)\, \varphi_\xi(\gamma)$$

$$\leq \underbrace{\sum N_{\bar{\alpha}\alpha}^{\gamma}\, d(\gamma)}_{= d(\alpha)^2}\ \underbrace{\varphi_\xi(\mathbb{1})}_{\leq \|\xi\|^2}$$

$$= d(\alpha)^2 \|\xi\|^2$$

Hence

$\pi \in \mathrm{Rep}_{\mathrm{adm}}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])$

$\Longrightarrow \|\pi(\alpha)\| \leq d(\alpha) \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}$.

(necessary condition).

☆ Later, we shall see

$$\begin{array}{ccc} \mathrm{Rep}_{\mathrm{adm}}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]) & \xleftarrow{\text{制限}} & \mathrm{Rep}(\mathrm{Tube}(\mathcal{C})) \\ \updownarrow & & \updownarrow \\ Z_s(\mathrm{ind}\text{-}\mathcal{C}) & \subset & Z(\mathrm{ind}\text{-}\mathcal{C}) \end{array}$$

Tube algebra

spherical objects

Drinfeld center

($\rightsquigarrow$ braided tensor category)

Defn. 2.21

$\mathcal{C}$: rigid (strict) C^*-tensor category.

$$C_u(\mathcal{C}) := C^*\text{-completion of } \mathbb{C}[\operatorname{Irr}\mathcal{C}]$$

w.r.t.

$$\|x\|_u := \sup_{\pi \text{ admissible}} \|\pi(x)\|$$

☆

- $\|x\|_u < +\infty$ since $\|\pi(\alpha)\| < d(\alpha)$ $\forall \pi$ admissible $\forall \alpha \in \operatorname{Irr}\mathcal{C}$.

- $\|x\|_u = 0 \Rightarrow x = 0$

  since $\pi_{\mathrm{reg}}(x) = 0$ implies $x = 0$.

  $$\left(x\, \delta_{\mathbb{1}} = \sum_{\pi} x_\pi\, \delta_\pi \right)$$

Ex. 2.22

$\mathcal{C} = \operatorname{Rep} SU(2)$

$$\begin{array}{ccc} \mathbb{C}[\operatorname{Irr}\mathcal{C}] & \simeq & \mathbb{C}[X] \\ \Psi & & \Psi \\ \pi_{1/2} & \longleftarrow & X \end{array} \qquad X^* = X$$

commutative

$$\rightsquigarrow C_u(\mathcal{C}) \cong C(\operatorname{sp}_{C_u(\mathcal{C})}(\underbrace{\pi_{1/2}}_{\text{self-adj.}}))$$

$\forall \rho : \mathbb{C}[\operatorname{Irr}\mathcal{C}] \to B(H_\rho)$ adm.

$$\|\rho(\pi_{1/2})\| \leq \underset{2}{\underset{\|}{d(\pi_{1/2})}}$$

$$\rightsquigarrow \operatorname{sp}_{C_u(\mathcal{C})}(\pi_{1/2}) \underset{\text{実際には} =}{\overset{}{\subset}} [-2, 2]$$

実際、

$$C_u(\mathcal{C}) \xrightarrow{*\text{-homo}} \mathcal{C}^*(\pi_{reg}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])) \curvearrowright \ell^2(\mathrm{Irr}\,\mathcal{C})$$

$\mathcal{C}^*(\pi_{reg}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])) = A$

ONB δ_{π_ν}, $\nu = 0, \frac{1}{2}, \dots$

であり

$$Sp_{C_u(\mathcal{C})}(\pi_{1/2}) \supset Sp_A(\pi_{reg}(\pi_{1/2}))$$

$$\pi_{reg}(\pi_{1/2}) = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & \ddots & \ddots \\ & & \ddots & \end{bmatrix}$$

$$= S + S^*$$

S: unilateral shift.

$\rightsquigarrow$ $A \subset C^*(S)$ Toeplitz alg

$$\begin{array}{ccccc} S+S^* & \in & A \subset C^*(S) & \ni & S \\ \downarrow & & \downarrow & & \downarrow \\ z+\bar{z} & \in & C(\mathbb{T}) & \ni & z \end{array}$$

modulo cpct.

$$Sp_{C(\mathbb{T})}(z+\bar{z}) = [-2, 2].$$

Ex. 2.23

$\mathcal{C} = \mathrm{Rep}(SU_q(2))$ $(q \neq 1)$

$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \simeq \mathbb{C}[x]$ commutative

$\pi_{1/2} \longmapsfrom x$

$\rightsquigarrow$ $C_u(\mathcal{C}) \cong C(Sp_{C_u(\mathcal{C})}(\pi_{1/2}))$

$\forall \rho : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \to B(H_\rho)$ adm.

$$\|\rho(\pi_{1/2})\| \leq d(\pi_{1/2}) = |q^{-1}+q| \underset{\neq}{>} 2$$

$\rightsquigarrow$ $Sp_{C_u(\mathcal{C})}(\pi_{1/2}) \subset [-|q^{-1}+q|, |q^{-1}+q|]$

π_{reg} さっきのギロン $\to \cup\ [-2, 2]$

絶対値は 2より大.

ちゃんと等しい… De Commer-Freslon-Yamashita

・Jones-Reznikoff.

まとめ

$$\begin{tikzcd}
\mathcal{C} \arrow[d, squiggly] \arrow[dd, bend right, "\text{CP性}"'] & \\
\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \arrow[d, squiggly] & \text{fusion alg.} \\
\mathrm{Rep}_{\mathrm{adm}}(\text{〃}) &
\end{tikzcd}$$

Section 3 で これを理解する方法を1つ紹介する。

Section 3. Tube algebras

§3.1 Defn.

$\mathcal{C}$: rigid, strict C^*-tensor category

morphism を忘れずに.

$\rightsquigarrow$ $\mathrm{Tube}(\mathcal{C})$ $*$-algebra.

$\cup$

$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$ corner $*$-subalgebra.

morphism を忘れた

$$\mathrm{Tube}(\mathcal{C}) := \bigoplus_{i,j\in \mathrm{Irr}\,\mathcal{C}} \mathrm{Tube}(\mathcal{C})_{i,j}$$

as a vector space.

dim $= \infty$ iff Irr $\mathcal{C}$ infinite.

$$\mathrm{Tube}(\mathcal{C})_{ij} := \bigoplus_{s\in \mathrm{Irr}\,\mathcal{C}} \cancel{\mathrm{Tube}}\ \mathrm{Mor}_{\mathcal{C}}(s\otimes j, i\otimes s)$$

arrow

as a vector space.

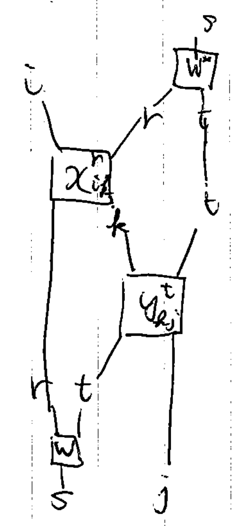
finite dimensional

Product

$$x = \sum_{i,k,s} x_{ik}^{s}, \quad y = \sum_{l,j} y_{lj}^{t}$$

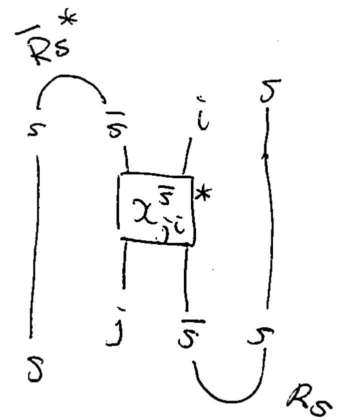
$$(x\cdot y)_{ij}^{s} := \sum_{\substack{k,r,t\in \mathrm{Irr}\,\mathcal{C} \\ w\in \mathrm{ONB}(s,r\otimes t)}} (1_i\otimes w^*)(x_{ik}^{r}\otimes 1_t)\cdot(1_r\otimes y_{kj}^{t})\cdot(w\otimes 1_j)$$

$$= \sum_{k,r,t,\,w}$$



*-operation.

$$(x^{\#})^{s}_{ij} = (\overline{R_s}^{*} \otimes 1_i \otimes 1_s) \cdot (1_s \otimes x^{\bar{s}\,*}_{ji} \otimes 1_s) \cdot (1_s \otimes 1_j \otimes R_s)$$

$=$ 

$(R_s, \overline{R_s})$ とり方によらない。

$\rightsquigarrow$ Tube($\mathcal{C}$) $*$-algebra ?

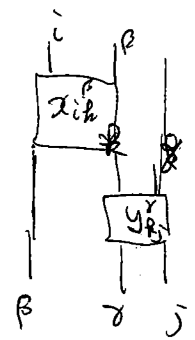
$$\widetilde{\mathrm{Tube}}(\mathcal{C}) := \bigoplus_{i,j \in \mathrm{Irr}\,\mathcal{C}} \widetilde{\mathrm{Tube}}(\mathcal{C})_{ij}$$

$$\widetilde{\mathrm{Tube}}(\mathcal{C})_{ij} := \bigoplus_{\alpha \in \mathcal{C}} \mathrm{Mor}(\alpha \otimes j, i \otimes \alpha)$$

☆ $\mathrm{Tube}(\mathcal{C})_{ij} \subset \widetilde{\mathrm{Tube}}(\mathcal{C})_{ij}$

Product.

$$(x \cdot y)^{\alpha}_{ij} = \sum_{\substack{k \in \mathrm{Irr}\,\mathcal{C} \\ \beta, \gamma \in \mathcal{C} \\ \beta \otimes \gamma = \alpha}} (x^{\beta}_{ik} \otimes 1_\gamma)(1_\beta \otimes y^{\gamma}_{kj})$$

$= \sum$ 

No.

$*$ - operation

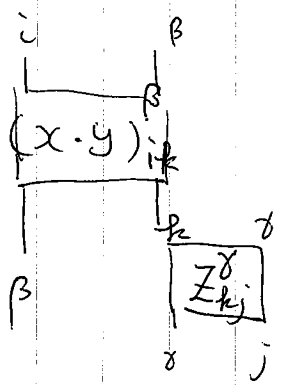
$$(x^\#)^\alpha_{ij} = (\overline{R}_\alpha^{\,*} \otimes 1_i \otimes 1_\alpha) \cdot (1_\alpha \otimes (x^{\bar\alpha}_{ji})^* \otimes 1_\alpha) \cdot (1_\alpha \otimes 1_j \otimes R_\alpha)$$

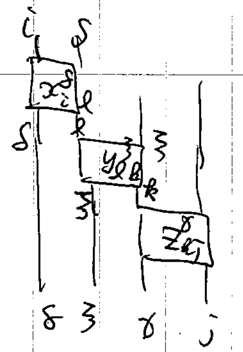
✩ We can show WMA $\bar{\bar\alpha} = \alpha$

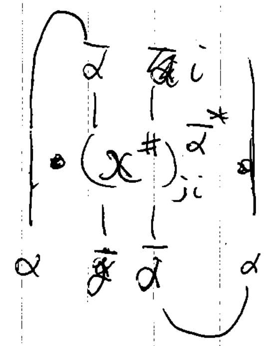
$$R_{\bar\alpha} = \overline{R}_\alpha \quad \overline{R}_{\bar\alpha} = R_\alpha$$

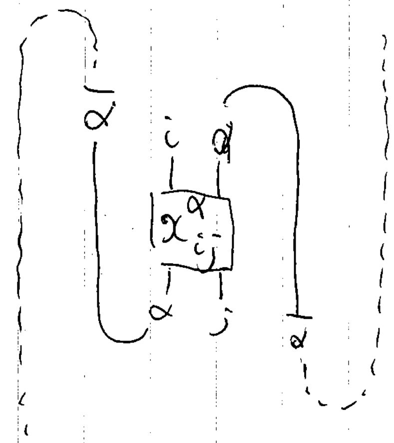
(Yamagami:)JOT. Frob duality...
spherical str.
due to
Müger

→ Tube($\mathcal{C}$) $*$-alg

$$[(x\cdot y)\cdot z]^\alpha_{ij} = \sum_{\substack{k,\ \beta\otimes\gamma \\ = \alpha}}$$ 

$$= \sum_{\delta\otimes\xi\otimes\gamma = \alpha}$$  $$= [x\cdot(y\cdot z)]^\alpha_{ij}$$

$$[(x^\#)^\#]^\alpha_{ij} =$$ 

$=$ 

$$= x^\alpha_{ij}$$

Now.

$$\Psi : \widetilde{\mathrm{Tube}}(\mathcal{C}) \longrightarrow \mathrm{Tube}(\mathcal{C})$$

$$\Psi(x_{ij}^{\alpha}) := \sum_{\substack{s<\alpha \\ w \in \mathrm{ONB}(s,\alpha)}} \cancel{w\otimes}(1_i \otimes w^{*})\, x_{sj}^{\alpha}\, (w \otimes 1_j)$$

$\uparrow$ $\mathrm{Mor}(s \otimes j, i \otimes s)$ におく.

☆ Ψ is ~~s~~ "projection map".

$$\Psi(x_{ij}^{s}) = x_{ij}^{s} \quad (s \in \mathrm{Irr}\,\mathcal{C})$$

☆ $\Psi(x \cdot y) = \Psi(x) \cdot \Psi(y)$

$$\Psi(x^{\#}) = \Psi(x)^{\#}.$$

$\rightsquigarrow$ $\mathrm{Tube}(\mathcal{C})$ $*$-algebra.

<u>Projections</u> p_i $(i \in \mathrm{Irr}\,\mathcal{C})$

Let $i \in \mathrm{Irr}\,\mathcal{C}$.

$p_i \in \mathrm{Tube}(\mathcal{C})$ is

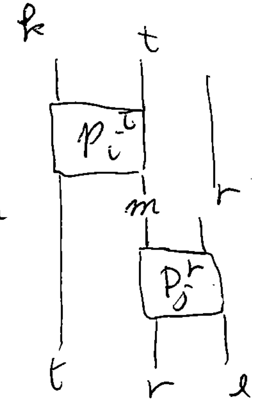
$$(p_i)_{k,l}^{s} = \delta_{s,\mathbb{1}}\, \delta_{k,i}\, \delta_{l,i}\, 1_i$$

$\cap$

$\mathrm{Mor}(s \otimes l, k \otimes s)$

☆ p_i is a projection. $i \in \mathrm{Irr}\,\mathcal{C}$

$$[p_i \cdot p_j]_{k,l}^{s} = \sum_{t,r,m}$$

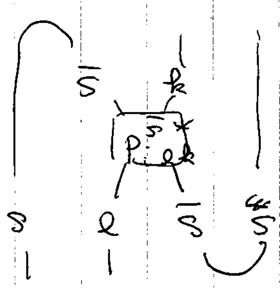


in $\widehat{\mathrm{Tube}}(\mathcal{C})$

$$= \delta_{k,i}\, \delta_{l,j}\, \delta_{s,\mathbb{1}}\, \delta_{i,j}$$

$$= [p_i]_{k,l}^{s}\, \delta_{i,j}$$

$\rightarrow$ $p_i \, p_j = \delta_{i,j} \, p_i$

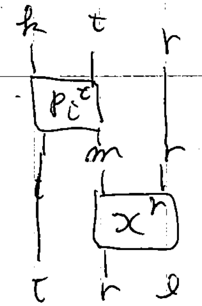
$\left[p_i^{\#} \right]^{s}_{k\ell} =$ 

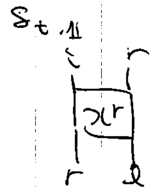
$= \delta_{s,1} \, \delta_{k,i} \, \delta_{\ell,i}$

$= \left[p_i \right]^{s}_{k\ell}$

$\rightarrow$ $p_i^{\#} = p_i$

☆ $p_i \, \mathrm{Tube}(\mathcal{C}) \, p_j = \mathrm{Tube}(\mathcal{C})_{i,j}$

$\left[p_i \, x \right]^{s}_{k,\ell} = \sum\limits_{t,r,m}$  in $\widetilde{\mathrm{Tube}}(\mathcal{C})$

$= \delta_{k,i}$ 

$\rightarrow$ $p_i \, x \in \mathrm{Tube}(\mathcal{C})_{i,\cdot}$

In particular

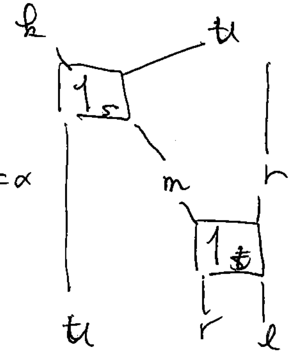
$\mathrm{Tube}(\mathcal{C})_{i,i} \subset \mathrm{Tube}(\mathcal{C})$

corner $*$-subalg.

$$\mathrm{Tube}(\mathcal{C})_{\mathbb{1},\mathbb{1}} = \bigoplus_{s\in\mathrm{Irr}\,\mathcal{C}} \mathrm{Mor}(s\otimes\mathbb{1},\ \mathbb{1}\otimes s)$$

$$= \bigoplus_{s\in\mathrm{Irr}\,\mathcal{C}} \mathbb{C}\,1_s$$

$$[1_s\cdot 1_t]^{\alpha}_{k,\ell} = \sum_{\substack{u\cdot r=\alpha\\ m}}$$



$$\delta_{m,\mathbb{1}} \qquad r=t,\ u=s$$

$$= \delta_{k,\mathbb{1}}\,\delta_{\ell,\mathbb{1}}\,1_{\underset{s}{\cancel{u}}\otimes t}$$

$$1_s\cdot 1_t = 1_{s\otimes t} \quad \text{in } \widetilde{\mathrm{Tube}}(\mathcal{C})$$

$$\xrightarrow{\Phi} \sum_u N_{s,t}^{u}\, y_u$$

$$[y_s^{\#}]^{t}_{\mathbb{1},\mathbb{1}} = \delta_{t,\bar{s}}$$

Prop. 3.1

$$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \xrightarrow{\sim} \mathrm{Tube}(\mathcal{C})_{\mathbb{1},\mathbb{1}} \qquad s\mapsto 1_s$$

as a $*$-alg

Hence

$$\mathrm{Rep}(\mathrm{Tube}(\mathcal{C})) \xrightarrow{\mathrm{rest}} \mathrm{Rep}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])$$

§3.2 Representations of Tube($\mathcal{C}$)

No. $\mathcal{C}$: rigid strict C^*-tensor category

$\mathrm{Mor}(\alpha, \beta)$ $\quad \alpha, \beta \in \mathcal{C}$.

is a fin dim Hilbert sp.

$\langle S, T \rangle := \mathrm{Tr}_\alpha(T^* S)$. $\quad S, T \in \mathrm{Mor}(\alpha,\beta)$

$\left(= \mathrm{Tr}_\beta(S T^*)\right)$ by Lem1.22

$\mathrm{ONB}(\alpha, \beta) := \{W_n\}_n$ an ONB

we fix

Lem. 3.2

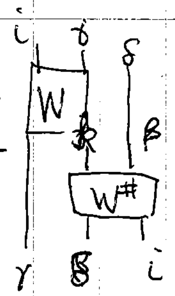
For $\forall i \in \mathrm{Irr}\,\mathcal{C}$ $\quad \forall \alpha \in \mathcal{C}$,

we have

$$\sum_{j \in \mathrm{Irr}\mathcal{C}} \sum_{W \in \mathrm{ONB}(\alpha j, i\alpha)} d(j)\, W \cdot W^{\#} = d(\alpha) p_i$$

in $\widetilde{\mathrm{Tube}}(\mathcal{C})$

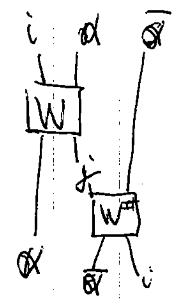
Proof.

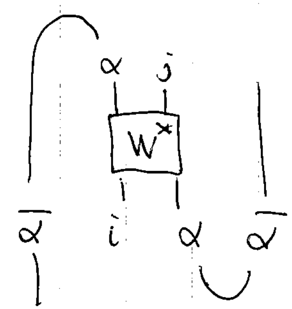
$[W \cdot W^{\#}]^{\beta}_{ii} = \sum$ 

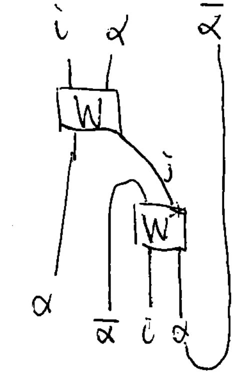
$\in p_i \widetilde{\mathrm{Tube}}(\mathcal{C}) p_i$

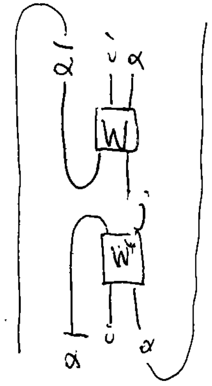
$\beta = \gamma \otimes \delta$.

$\delta_{k,j}$

$= \sum_{\beta = \gamma\otimes\delta}$  —①

$[W^{\#}]^{\bar\alpha}_{ji} =$ 

① = 

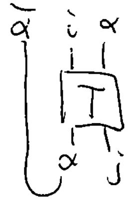
= 

— ②

Now recall

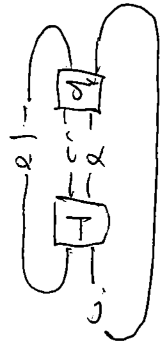
$$\mathrm{Mor}(\alpha\otimes j, i\otimes\alpha) \xrightarrow{F} \mathrm{Mor}(j, \bar\alpha\otimes i\otimes\alpha)$$

$$\begin{array}{ccc} \cup & & \cup \\ T & \longmapsto & \sqrt{d(j)}\ \cancel{R_\alpha}\ (1_{\bar\alpha}\otimes T)(R_\alpha\otimes 1_j) \end{array}$$

$\sqrt{d(j)}$ 

is unitary.

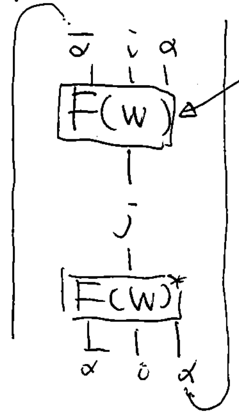
$$\langle F(T), F(S)\rangle = \mathrm{Tr}_j\left(F(S)^* F(T)\right)$$

$= d(j)$ 

$= d(j)\ \mathrm{Tr}_{\alpha\otimes j}(\cancel{T} S^*_T)$

$= d(j)\ \mathrm{Tr}\cancel{_{i\otimes\alpha}}\langle T, S\rangle.$

~~Let V_T~~

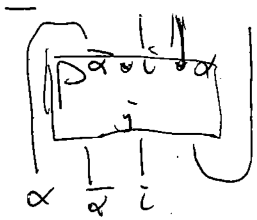
② = 

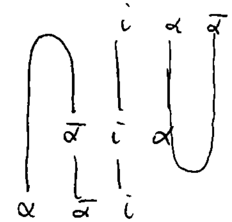
ONB of $\mathrm{Mor}(j, \bar\alpha\otimes i\otimes\alpha)$

$\therefore\ F(W)^*F(W) = \dfrac{1}{d(j)}$

$\sqrt{d(j)}\,F(W)$ isometry

$\sum_W d(j)$ —〃—

$=$ 

$\xrightarrow{\sum_j}$ 

Ψ $\longrightarrow d(\alpha)P_i$

∎

Lem. 3.3

Any repn of Tube($\mathcal{C}$) on a preHilbert sp. is bounded: $\forall \pi : \mathrm{Tube}(\mathcal{C}) \to \mathcal{L}(H)$ we have

$$\|\pi(T)\| \leq \sqrt{\frac{d(\alpha)}{d(j)}}\,\|T\|_2 \leq d(\alpha)\|T\|$$

for $\forall T \in \mathrm{Mor}(\alpha j, i\alpha)$

$\forall i, j \in \mathrm{Irr}\,\mathcal{C}$, $\alpha \in \mathrm{Irr}\,\mathcal{C}$

☆ H : pre-Hilb. sp.

$\mathcal{L}(H) := \{$ linear maps on $H \}$

$\cup$

$B(H)$

$\pi : \overset{*\text{-alg}}{A} \longrightarrow \mathcal{L}(H)$ is a <u>repn</u>

iff linear, multiplicative &

$\langle \pi(x)\xi, \eta\rangle = \langle \xi, \pi(x^*)\eta\rangle$ $\forall \xi, \eta \in H$.

Proof

Let $\pi : \mathrm{Tube}(\mathcal{C}) \to \mathcal{L}(H)$ repn given

By the previous lemma, we have

$$\sum_{j, W} \langle d(j)\,\pi(W)\,\pi(W^\#)\,\xi, \xi\rangle = d(\alpha)\|P_i^{\pi}\xi\|^2 \quad (\text{proj.})$$

$$\| \quad d(j)\,\|\pi(W^\#)\xi\|^2$$

$\rightsquigarrow$ $\pi(W^\#)$ & hence $\pi(W)$ bdd.

Let $T \in \mathrm{Mor}(\alpha j, i\alpha)$ put $\dfrac{T}{\|T\|_2}$ for W

$$d(j)\,\frac{\|\pi(T)\xi\|^2}{\|T\|_2^2} \leq d(\alpha)\|\xi\|^2$$

$\|T\|_2 = \mathrm{Tr}_{\alpha j}(T^*T)^{1/2} \leq \|T\|\,\dfrac{d(\alpha)^{1/2}}{d(j)^{1/2}}$

$\to$ $\|\pi(T)\xi\| \leq \dfrac{d(\alpha)^{1/2}}{d(j)^{1/2}}\,\|T\|_2\,\|\xi\|$

$\rightsquigarrow$ $\|\pi(T)\| \leq \dfrac{d(\alpha)^{1/2}}{d(j)^{1/2}}\,\|T\|_2 \leq d(\alpha)\,\|T\|$

Defn. 3.4

$$\boxed{\begin{array}{l} C^*(\mathrm{Tube}(\mathcal{C})) \text{ is the } C^*\text{-completion} \\ \text{of } \mathrm{Tube}(\mathcal{C}) \text{ w.r.t. the } C^*\text{-norm} \\ \|x\|_u := \sup\limits_{\forall \pi \text{ repn of } \mathrm{Tube}(\mathcal{C})} \|\pi(x)\| \end{array}}$$

☆ The prev. lemma shows

$$\|x\|_u < +\infty$$

☆ $\|x\|_u$ is ~~as~~ indeed a norm

as shown ~~below~~ from now.

Ex. (Regular repn.)

$$\begin{array}{ccccl} \tau : & \mathrm{Tube}(\mathcal{C}) & \longrightarrow & \mathbb{C} & \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \subset \mathrm{Mor}(i,i) = \mathbb{C}1_i \\ & x & \longmapsto & \sum_i d(i) x_{ii}^{1} & \\ & & & \| & \\ & & & \sum_i \mathrm{Tr}_i(x_{ii}^{1}) & \end{array}$$

is a faithful tracial positive functional.

$x \in \mathrm{Mor}(i\alpha, \alpha j)$ $\quad i, j, k, l \in \mathrm{Irr}\,\mathcal{C}$

$y \in \mathrm{Mor}(k\beta, \beta l)$ $\quad \alpha, \beta \in \mathrm{Irr}\,\mathcal{C}$

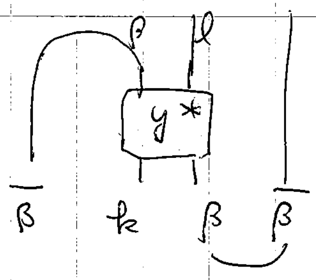
Then

$$\tau(xy^*) = \delta_{\alpha,\beta}\, \delta_{ik}\, \delta_{j,l} \frac{1}{d(\alpha)} \mathrm{Tr}_{i\alpha}(xy^*)$$

$\|$ naturally

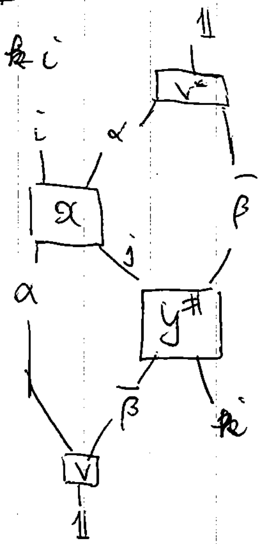
$$\tau(y^*x) = \delta_{\alpha,\beta}\, \delta_{ik}\, \delta_{j,l} \frac{1}{d(\alpha)} \mathrm{Tr}_{\alpha j}(y^*x)$$

$y^{\#} =$



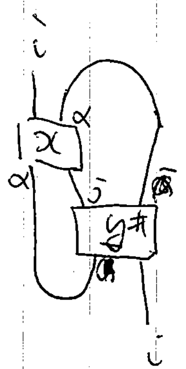
$\left[x \cdot y^{\#} \right]_{i,k\,i}^{1\!\!1}$ (p_i, p_k)

$= \delta_{j,l}\, \delta_{i,k}$

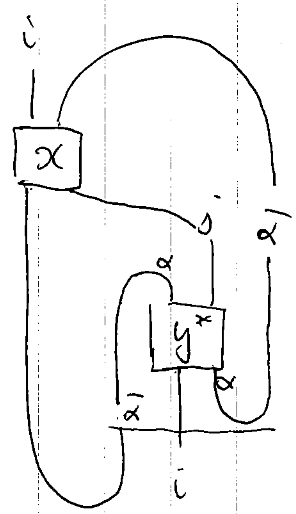


$V = \bar{R}_\alpha \dfrac{1}{d(\alpha)^{1/2}}$

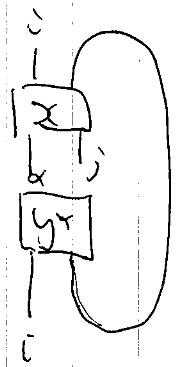
$= \delta_{j,l}\, \delta_{\alpha,\beta}\, \delta_{i,k} \cdot \dfrac{1}{d(\alpha)}$



$= \delta_{\alpha,\beta}\, \delta_{i,k}\, \delta_{j,l} \dfrac{1}{d(\alpha)}$



$= \delta_{\alpha,\beta}\, \delta_{i,k}\, \delta_{j,l} \dfrac{1}{d(\alpha)}$



$\to \tau(xy^*) = \delta_{\alpha,\beta}\, \delta_{i,k}\, \delta_{j,l} \dfrac{1}{d(\alpha)} \mathrm{Tr}_{i\otimes\alpha}(xy^*)$

$\rightsquigarrow \ \forall x = \sum_{\alpha, i, j} x^{\alpha}_{ij}$

$$\tau(x^* x) = \sum_{\alpha, i, j} \frac{1}{d(\alpha)} \mathrm{Tr}_{\alpha \circ j}(x^{\alpha *}_{ij} x^{\alpha}_{ij})$$

Hence τ is faithful.

$\rightsquigarrow$ GNS repn

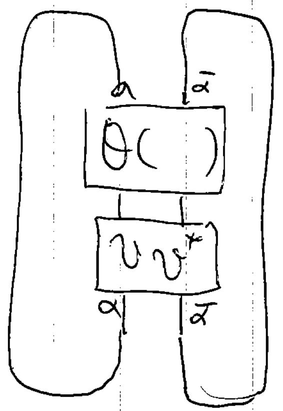
$\pi_{reg} \curvearrowright L^2(\mathrm{Tube}(\mathcal{C}))$

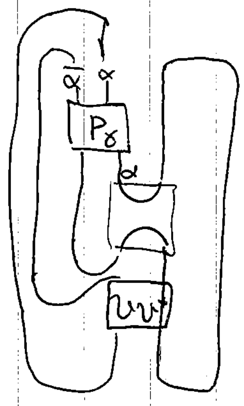
Now let $\varphi : \mathrm{Irr}\,\mathcal{C} \to \mathbb{C}$

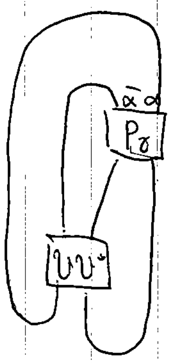
$\alpha \in \mathcal{C}$

$\langle \Theta^{\varphi}_{\alpha,\bar{\alpha}} (\bar{R}_\alpha \bar{R}_\alpha^*) v, v \rangle_{\mathrm{End}(\bar{\alpha} \otimes \alpha)}$

$= \mathrm{Tr}_{\alpha \otimes \bar{\alpha}} \left(\Theta^{\varphi}_{\alpha.\bar{\alpha}} (\bar{R}_\alpha \bar{R}_\alpha^*) v v^* \right)$

$=$ 

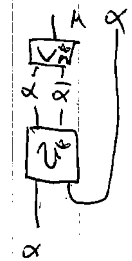
$= \sum_\gamma \varphi(\gamma)$ 

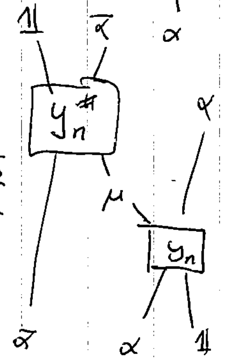
$= \sum_\gamma \varphi(\gamma)$  $- (1)$

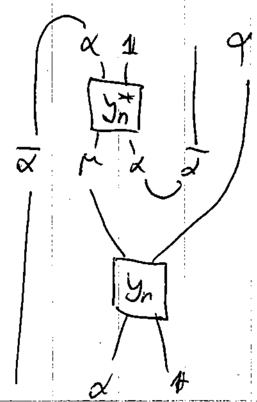
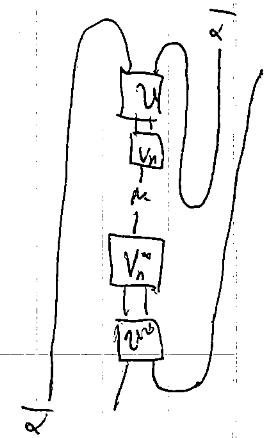
We show

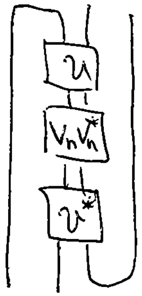
$(1) = \omega_{\varphi^{op}} \left(\sum_n y_n^{\#} \cdot y_n \right)$.

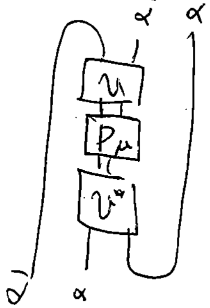
$y_n \in (y^{\alpha}_{\mu.\mathbb{1}})_\mu \in \widetilde{\mathrm{Tube}(\mathcal{C})}^{\alpha}_{\cdot\cdot\mathbb{1}}$

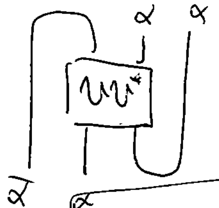
& ~~$y^{\alpha}_{\mu.\mathbb{1}}$~~  $=$  $\in \widehat{\mathrm{Tube}(\mathcal{C})}^{\alpha}_{\mu,\mathbb{1}}$

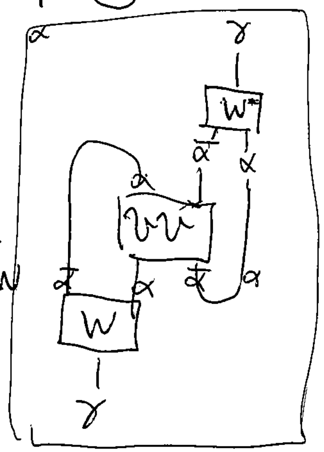
$y_n^{\#} \cdot y_n = \sum_\mu$ 

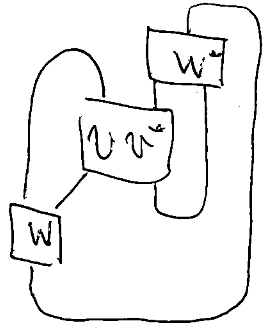
$= \sum_\mu$  $= \sum_\mu$ 

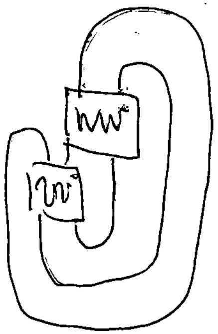
$= \sum_{\mu}$ 

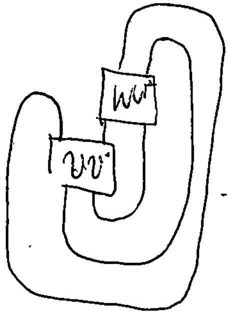
$\to \sum_n y_n^{\#} \cdot y_n = \sum_{\mu}$ 

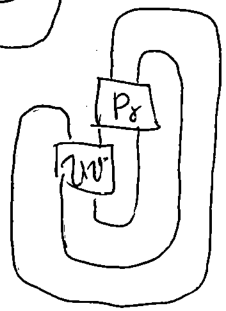
$=$  $\in \widetilde{\mathrm{Tube}}(\mathcal{C})_{1\!\!1\, 1\!\!1}^{\bar\alpha \otimes \alpha}$

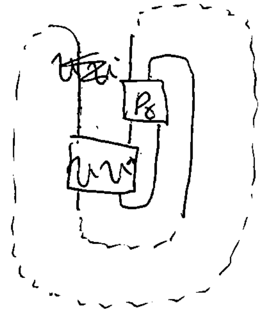
$\Phi(\sum_n y_n^{\#} \cdot y_n) = \sum_{\gamma . W}$  $= \lambda 1_\gamma$

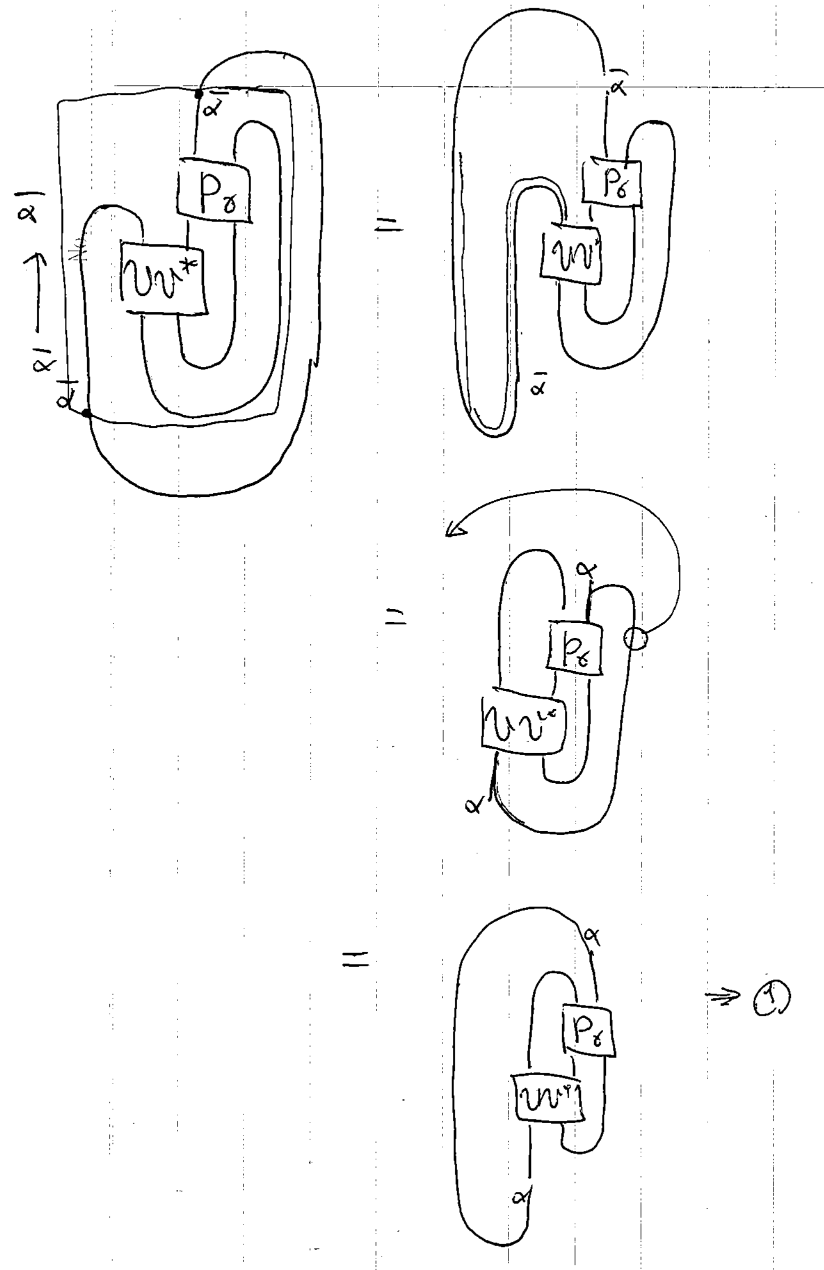
 $= \lambda d(\gamma)$

$=$ 

$\to \lambda = \frac{1}{d(\gamma)}$ 

$\Phi(\sum_n y_n^{\#} \cdot y_n) = \sum_\gamma \frac{1}{d(\gamma)}$  1_γ

$\omega_{\varphi^{op}}(\Phi(\sum_n y_n^{\#} \cdot y_n)) = \sum_\gamma \varphi(\bar\gamma)$  ②



Hence.

$$\langle \theta^{\varphi}_{\alpha,\bar\alpha}(\bar R_\alpha \bar R_\alpha^*) v, v \rangle$$

$$= \omega_{\varphi^{op}}\Big(\sum_n y_n^\# \cdot y_n\Big)$$

$$\omega_{\varphi^{op}} \geqq 0 \Rightarrow \theta^{\varphi}_{\alpha.\bar\alpha}(\bar R_\alpha \bar R_\alpha^*) \geqq 0$$

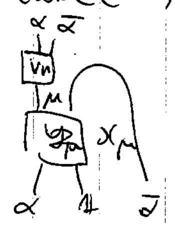
$$\Rightarrow \varphi \text{ c.p. mult.}$$

Conversely. suppose φ c.p. mult.

$$\Big(\Rightarrow \omega_{\varphi^{op}}\big(\textstyle\sum_n y_n^\# \cdot y_n\big) \geqq 0. \Big)$$

$x_\mu \in \mathrm{Tube}(\mathcal{C})^{\alpha}_{\mu.1}$

$v_{n,\mu} :=$



$$v_{n.\mu} \rightsquigarrow y^{\alpha,m}_{\mu.1} = \delta_{m.n}\, x_\mu.$$

$$\langle \theta^{\varphi}_{\alpha.\bar\alpha}(\bar R_\alpha \bar R_\alpha^*) v, v \rangle = \omega_{\varphi^{op}}(x_\mu^* x_\mu)$$

Hence

<u>Thm.</u>

$\varphi : \operatorname{Irr} \mathcal{C} \to \mathbb{C}$

TFAE.

(1) φ c.p. mult

(2) φ^{op} — " —

(3) $\widetilde{\omega}_\varphi$ ~~c.p. mult~~ positive on $\mathrm{Tube}(\mathcal{C})$

<u>Proof</u>

(1) $\Leftarrow$) $\omega_{\varphi^{op}}$ positive. Done

$$\omega_{\varphi^{op}} = \omega_\varphi \circ K$$

K: antimultiplicative auto of $\mathrm{Tube}(\mathcal{C})$, $*$-preserving

$K : \mathrm{Tube}(\mathcal{C}) \to \mathrm{Tube}(\mathcal{C})$.

$$\underset{\mathrm{Mor}(i\alpha, \alpha j)}{K(\underset{\cap}{x})} := \underset{\mathrm{Mor}(\bar{j}\bar{\alpha}, \bar{\alpha}\bar{i})}{\underset{\cap}{x^\vee}}$$

This result leads up

<u>Thm.</u>

(1) $\forall \pi : \mathrm{Tube}(\mathcal{C}) \to B(H)$ $*$-repn.

Then $\pi\restriction_{\mathbb{C}[\operatorname{Irr}\mathcal{C}]}$ is admissible

(2) $\forall$ $\sigma : \mathbb{C}[\operatorname{Irr}\mathcal{C}] \to B(K)$ admissible

$\exists$ $\pi : \mathrm{Tube}(\mathcal{C}) \to B(H)$ $*$-repn, $\pi(p_1) = 1_H$

s.t. $\pi\restriction_{\mathbb{C}[\operatorname{Irr}\mathcal{C}]} \underset{\text{unitary}}{\cong} \sigma$

Proof.

(1) $\pi : \mathrm{Tube}(\mathcal{C}) \longrightarrow B(H)$

$\xi \in H$.

Set
$$\varphi(\alpha) := \frac{1}{d(\alpha)} \langle \pi(\alpha)\xi, \xi \rangle \quad \alpha \in \mathrm{Irr}\,\mathcal{C}$$
WMA $\xi = \pi(p_{\mathbb{1}})\xi$.

&

$$\omega_\varphi : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow \mathbb{C}$$

$$\omega_\varphi(\alpha) := d(\alpha)\varphi(\alpha) = \langle \pi(\alpha)\xi, \xi \rangle$$

i.e.
$$\omega_\varphi(x) = \langle \pi(x)\xi, \xi \rangle \quad \forall x \in \mathbb{C}[\mathrm{Irr}\,\mathcal{C}]$$

$$\rightarrow \widetilde{\omega_\varphi}(x) = \langle \pi(x)\xi, \xi \rangle \quad \forall x \in \mathrm{Tube}(\mathcal{C})$$

positive.

$\rightarrow \varphi$ cp-mult. $\rightarrow \pi\restriction_{\mathbb{C}[\mathrm{Irr}\,\mathcal{C}]}$ adm.

(2) $\sigma : \mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \longrightarrow B(K)$ adm.

WMA $\exists$ cyclic vector ξ ($\sigma(p_{\mathbb{1}})\xi = \xi$)

$$\varphi(\alpha) := \frac{1}{d(\alpha)} \langle \sigma(\alpha)\xi, \xi \rangle \quad \forall \alpha \in \mathrm{Irr}\,\mathcal{C}$$

cp-mult.

$$\omega_\varphi(\alpha) = \langle \sigma(\alpha)\xi, \xi \rangle$$

$$\widetilde{\omega_\varphi}(x) = \langle \sigma(p_{\mathbb{1}} x p_{\mathbb{1}})\xi, \xi \rangle \quad \forall x \in \mathrm{Tube}(\mathcal{C})$$

$\rightarrow$ positive

$$\pi_{\widetilde{\omega_\varphi}} : \mathrm{Tube}(\mathcal{C}) \longrightarrow B(H_{\widetilde{\omega_\varphi}}) \quad \text{GNS}.$$

$$H_{\widetilde{\omega_\varphi}} = \overline{\mathrm{Tube}(\mathcal{C})\xi_{\widetilde{\omega_\varphi}}} = \overline{\mathrm{Tube}(\mathcal{C})_{\cdot\mathbb{1}}}$$

$$\sigma \underset{\pi_{\widetilde{\omega_\varphi}}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])}{\cong} \overline{\mathrm{Tube}(\mathcal{C})_{\mathbb{1}\mathbb{1}}} \oplus \overline{\mathrm{Tube}(\mathcal{C})_{k\mathbb{1}}}^{\perp}$$

0

Cor.

$$\mathbb{C}[\mathrm{Irr}\,\mathcal{C}] \hookrightarrow \mathrm{Tube}(\mathcal{C})$$

extends to

$$\mathrm{Cu}(\mathcal{C}) \hookrightarrow C^*(\mathrm{Tube}(\mathcal{C}))$$

Section 4. Drinfeld center $Z(\mathrm{ind}-\mathcal{C})$

$\mathcal{T}$: C^*-tensor category strict

(but we donot assume $\mathcal{T}$ is rigid)

Defn.

Let $Z \in \mathcal{T}$.

A unitary half-braiding on Z is c

s.t.

(1) c is a family $(c_\alpha)_{\alpha \in \mathcal{T}}$

$$c_\alpha : \alpha \otimes Z \longrightarrow Z \otimes \alpha \quad \text{unitary}$$

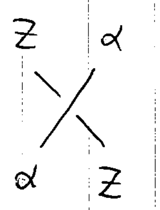
(2) c_α is natural w.r.t. α:

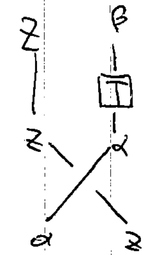
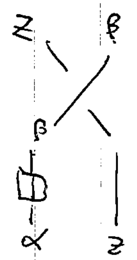
If $\alpha \xrightarrow{T} \beta$ then

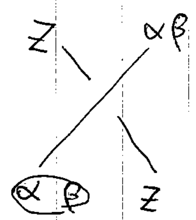
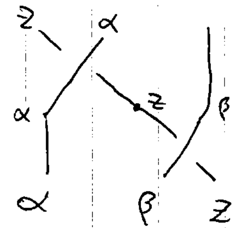
$$(1_Z \otimes T)\, c_\alpha = c_\beta (T \otimes 1_Z)$$

(3) $c_{\mathbb{1}} = \mathrm{id}_Z : \mathbb{1} \otimes Z \to Z \otimes \mathbb{1}$.

(4) $c_{\alpha \otimes \beta} = (c_\alpha \otimes 1_\beta) \cdot (1_\alpha \otimes c_\beta)$

☆ $c_\alpha =$ 

(2)  $=$ 

(4)  $=$ 

☆ $Z(\mathcal{T}) := \{(z, c)\}$. $\nearrow$ braiding $\sigma_{(z_1,c_1),(z_2,c_2)} = (c_2)_{z_1}$

$(z_1, c_1) \otimes (z_2, c_2) = (z_1 \otimes z_2, (1_{z_1} \otimes c_2)(c_1 \otimes 1_{z_2}))$

$(z_1, c_1) \xrightarrow{T} (z_2, c_2)$ is $z_1 \xrightarrow{T} z_2$ & $(T \otimes 1)\, c_1 = c_2 (1 \otimes T)$

Ind-category

$\mathcal{C}$: C^*-tensor category strict
rigid.

which admits finite direct sum

$\mathcal{C} \subset \text{Ind-}\mathcal{C}$ ← arbitrarily number of direct sum is O.K.

C^*-tensor category strict

but not rigid.

(rigid part $= \mathcal{C}$).

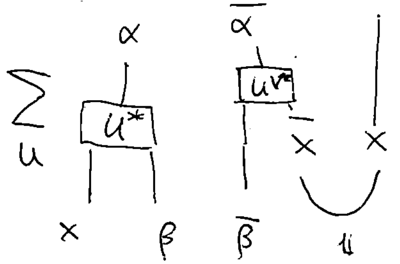
Ex.

$\text{Ind-}\mathcal{C} \ni Z_{reg} := \bigoplus_{\alpha \in \text{Irr}\,\mathcal{C}} \alpha \otimes \bar{\alpha}$

which has a unitary half-braiding.

$X \in \mathcal{C}$

$X \otimes \beta \otimes \bar{\beta} \xrightarrow{c_{X,\alpha,\beta}} \alpha \otimes \bar{\alpha} \otimes X$

$= \sqrt{\dfrac{d(\beta)}{d(\alpha)}} \sum_{u}$ 

$u : \alpha \to X \otimes \beta$ ONB

$u^\vee : \bar{\beta} \otimes \bar{X} \to \bar{\alpha}$

$\to c_X := (c_{X,\alpha\beta})_{\alpha,\beta}$

$: X \otimes Z \to Z \otimes X$

is a unitary half-braiding.

$\mathcal{Z}(\text{ind-}\mathcal{C})$ & $\mathrm{Rep}(\mathrm{Tube}(\mathcal{C}))$

Let $(Z, c) \in \mathcal{Z}(\text{ind-}\mathcal{C})$

$$H_i := \mathrm{Mor}(i, Z) \quad i \in \mathrm{Irr}\,\mathcal{C}$$
$\mathrm{Mor}(Z, i)$

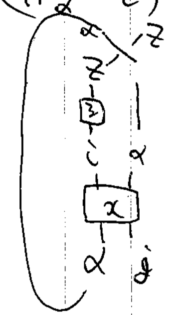
&

$$H := \bigoplus_{i \in \mathrm{Irr}} H_i$$

Set $\forall \xi \in H_i$, $\forall x \in \mathrm{Mor}(i\alpha, \alpha j)$

$$\xi \cdot x := (\mathrm{Tr}_\alpha \otimes \iota)(c_\alpha^* (\xi \otimes 1) x)$$

$$= \quad \in H_j$$



$\rightsquigarrow$ H is a right $\mathrm{Tube}(\mathcal{C})$-module.

Conversely let H : ~~the~~ a right $\mathrm{Tube}(\mathcal{C})$-module

Let $Z \in \text{ind-}\mathcal{C}$ s.t.

$$\mathrm{Mor}(Z, i) = H p_i$$

$Z_i :=$ isotypical part of Z w.r.t. $i \in \mathrm{Irr}\,\mathcal{C}$.

$\rightsquigarrow$ $Z = \bigoplus_i Z_i$

Now. for $x \in \mathrm{Mor}(i\alpha, \alpha j)$

$$\langle \underset{H p_i}{\xi} \cdot \underset{H p_j}{\eta} \rangle := \langle \xi x, \eta \rangle$$

$\rightsquigarrow$ $\exists\, \sigma_{\alpha, ij} \in \mathrm{Mor}(Z_i \alpha, \alpha Z_j)$ $\qquad (\xi \otimes 1) x (1 \otimes \eta^*)_{\alpha \otimes Z}$

$$\mathrm{Tr}_{\alpha_j}(\underbrace{(1 \otimes \eta^*) \sigma_{\alpha, ij}^* (\xi \otimes 1)}_{\alpha \otimes j \;\longleftarrow\; i \otimes \alpha} x) = \langle \xi x, \eta \rangle$$

$$\|x\|_u \leq \frac{d(\alpha)^{1/2}}{d(j)^{1/2}} \|x\|_2$$

$$
\begin{tikzcd}
\mathrm{Rep}(\mathrm{Tube}(\mathcal{C})) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{rest}"] & \cancel{\mathrm{Rep}}\ Z(\mathrm{ind}\text{-}\mathcal{C}) \\
\mathrm{Rep}(\mathbb{C}[\mathrm{Irr}\,\mathcal{C}])_{\mathrm{adm}} &
\end{tikzcd}
$$

as a C^*- ~~t…~~ category.