Martin 境界入門

1. Markov chains
2. Harmonic functions
3. Martin boundaries
4. Convergence to the boundary

Reference

Dynkin : Boundary theory of Markov Processes
Section 4 の証明はこれに従った.
大変良くまとまっている.

Sawyer : Martin boundaries & random walks
"Convergence to the boundary" が
証明されてそうで実はされていない
ので注意. (P.15. Section 5 の冒頭の
仮定がある).

Woess : Random walks & infinite graphs & groups
極小境界の理論は負数の部分
ゆえさすがに証明されていない.
曰く「Dynkin がベスト」

Kemeny-Snell-Knapp : Denumerable Markov chains
"Convergence to the boundary"
証明有. (しかし読み易いかというと…?)

Revuz : Markov chains

Discrete に限らないやり方.
あんまり読んでいないが. ちゃんと
書いてあると思う.

Spitzer : Principles of Random Walks

$\mathbb{Z}^d$ のケースに限るが. 沢山
計算してくれている.

本来は

Martin - Doob - Hunt

も読み込んでおくべきだが. 余り読まなかった.
(むずかしかった)

他. 論説で日本語の

国田 - 渡辺 : Markov chain と Martin 境界

もよいだろう

2017年 11月6日 - 10日 @東大

Section 1. Markov chains

§1.1 Markov chains

<u>Notation</u>

$$\mathbb{N} = \{0, 1, 2, \cdots\}$$

$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geqq 0\}$$

$$X = \text{denumerable set}$$

$$\# X \leqq \# \mathbb{N}$$

<u>Defn</u> 1.1

A X-by-X matrix $P = (p(x,y))_{x,y \in X}$

is a <u>transition matrix</u>

when

(1) $p(x,y) \geqq 0 \quad \forall x, y \in X$

(2) $\displaystyle\sum_{y \in X} p(x,y) = 1 \quad \forall x \in X$.

We call (X, P) a <u>Markov chain</u>.

<u>Rem</u>. 1.2.

- $p(x,y)$ the transition probability from x to y in one step
- We sometimes consider P with (1) & (2)'

(2)' $\displaystyle\sum_{y \in X} p(x,y) \leqq 1 \quad \forall x \in X$.

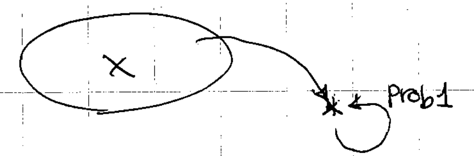
This P extends to $\widehat{P}$ as follows:

$$\widetilde{X} := X \cup \{*\}$$

$$\widehat{P} := \left[\begin{array}{c|c} P & \mathbb{1} - P\mathbb{1} \\ \hline 0 & 1 \end{array}\right]$$

where $\mathbb{1}(x) = 1 \quad \forall x \in X$.

$$(\mathbb{1} - P\mathbb{1})(x) = 1 - \sum_y p(x,y) \geqq 0.$$



graveyard or cemetery pt.

- $P^n = (p_n(x,y))_{x,y\in X}$ the n-th power of trans. matrix P

$$p_n(x,y) = \sum_{x_1,\dots,x_{n-1}\in X} p(x,x_1)\,p(x_1,x_2)\cdots p(x_{n-1},y).$$

the transition prob from x to y in the n steps

<u>NOTE</u>

$$\sum_{y\in X} p_n(x,y) = 1.$$

$$\uparrow$$

$$P^n\mathbb{1} = P^{n-1}P\mathbb{1} = P^{n-1}\mathbb{1} = \cdots = \mathbb{1}.$$

- $P^0 := I = (\delta_{x,y})_{x,y\in X}$. $p_0(x,y) := \delta_{x,y}$

<u>Ex.</u> 1.3

Γ : discrete grp

$$\mathrm{Prob}(\Gamma) := \{\mu \in \ell_1\Gamma \mid \mu(s)\geq 0\ \forall s\in\Gamma,\ \sum_{s\in\Gamma}\mu(s) = 1\}$$

$\mathrm{Prob}(\Gamma) \ni \mu \rightsquigarrow P_\mu = (p_\mu(x,y))_{x,y}$

trans. matrix.

$p_\mu(x,y) = \mu(x^{-1}y) = p_\mu(tx,ty)$

$x \longrightarrow y = xs$ $\quad p_\mu(x,xs) = \mu(s)$

For $\mu,\nu\in\mathrm{Prob}(\Gamma)$, define $\mu*\nu\in\mathrm{Prob}(\Gamma)$

$$\mu*\nu(s) := \sum_{t\in\Gamma}\mu(t)\,\nu(t^{-1}s)$$

$$= \sum_{\substack{x,y\in\Gamma\\ xy=s}} \mu(x)\,\nu(y)$$

$G\times G \to G$

$\mu\times\nu \longmapsto \mu*\nu$

push

<u>NOTE</u>

- $P_{\mu*\nu} = P_\mu P_\nu$

$$\sum_y p_\mu(x,y)\,p_\nu(y,z) = \sum_y \mu(x^{-1}y)\,\nu(y^{-1}z)$$

$$= \sum_y \mu(y)\,\nu(y^{-1}x^{-1}z) = \mu*\nu(x^{-1}z)$$

$$\parallel$$

$$p_{\mu*\nu}(x,z).$$

- $\mathrm{supp}\,\mu := \{t\in\Gamma \mid \mu(t)>0\}$

$\mathrm{supp}(\mu*\nu) = \mathrm{supp}\,\mu\cdot\mathrm{supp}\,\nu$

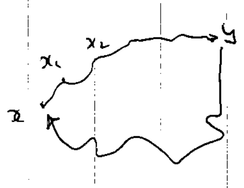
$= \{st \mid s\in\mathrm{supp}\,\mu,\ t\in\mathrm{supp}\,\nu\}$

§ 1.2. Irreducibility, recurrence, transience

Defn. 1.4

(X, P) is irreducible

if $\forall x, \forall y \in X$ $\exists n \in \mathbb{N}$ s.t. $p_n(x,y) \gneq 0$



i.e. $\exists x_1, \dots x_{n-1} \in X$ $p(x,x_1), \dots, p(x_{n-1},y) > 0$.

Basic Assumption.

We assume (X, P) irreducible unless otherwise noted.

Ex. 1.5

$\mu \in Prob(\Gamma)$

P_μ irreducible $\Leftrightarrow \bigcup_{n \geq 0} supp(\mu^{*n}) = \Gamma$ (i.e. μ is non-degenerate)

$\mu^{*0} = \delta_e$

$p_n(x,y) = \mu^{*n}(x^{-1}y)$

Green kernel

Defn. 1.6

(X, P) Markov chain

Set

$$G(x,y) := \sum_{n=0}^{\infty} p_n(x,y) \qquad x, y \in X$$

$$= \delta_{x,y} + p_1(x,y) + \cdots \leq +\infty.$$

$G(x,y)$ is called the Green kernel (or potential)

☆ Let $G := (G(x,y))_{x,y \in X}$.

$$= I + P + P^2 + \cdots$$

(geometric series)

$\to I + PG = G = I + GP$.

Lem. 1.7

(X, P) irr. Markov chain.

Let $x_1, x_2, y_1, y_2 \in X$.

Take $k, l \in \mathbb{N}$ such that

$$p_k(x_2, x_1) > 0$$

$$p_l(y_1, y_2) > 0.$$

Then

$$G(x_1, y_1) \leqq \frac{1}{p_k(x_2, x_1)\, p_l(y_1, y_2)}\, G(x_2, y_2)$$

☆ This implies

$\forall x, z, \forall y \ \exists C_{x,z}, C'_{x,z} > 0$. (depends on x & z)

$$C_{x,z}\, G(z, y) \leqq G(x, y) \leqq C'_{x,z}\, G(z, y).$$

Proof.

$$p_k(x_2, x_1)\, p_n(x_1, y_1)\, p_l(y_1, y_2) \leqq p_{k+n+l}(x_2, y_2)$$

Σ sum with n

$$\text{—〃—} \quad G(x_1, y_1) \quad \text{—〃—} \quad \leqq G(x_2, y_2)$$

Defn. 1.8

(X, P) is

(1) ~~transient~~ recurrent $\overset{\text{defn}}{\Longleftrightarrow}$ $G(x, y) = +\infty$ for $\exists x, y \in X$

$\Longleftrightarrow$ $G(x, y) = +\infty$ for $\forall x, y \in X$

(2) transient $\overset{\text{defn}}{\Longleftrightarrow}$ $G(x, y) < +\infty$ for $\exists x, y \in X$

$\overset{\cancel{\text{defn}}}{\Longleftrightarrow}$ $G(x, y) < +\infty$ for $\forall x, y \in X$

Ex. 1.9

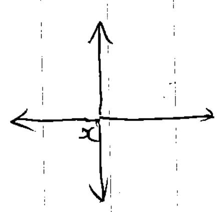
$\Gamma = \mathbb{Z}^d \quad (d = 1, 2, \ldots)$

$\mathbb{Z}^d = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_d$

$\varepsilon_k = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow k\text{-th}$

Let $\mu \in \mathrm{Prob}(\mathbb{Z}^d)$

$\mu(\pm\varepsilon_k) = \frac{1}{2d}$ SRW simple random walk



equal probability

$\to (\mathbb{Z}^d, P_\mu)$. irr. Markov chain

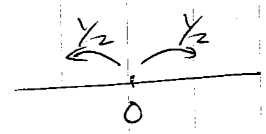
Thm. ~~2.0~~ 1.10 (Pólya)

> SRW on $\mathbb{Z}^d$ is
> - recurrent if $d = 1, 2$
> - transient if $d = 3, 4, \ldots$

Proof. 1. (Combinatorics)

d = 1. We need to check

$G(0,0) = +\infty$.



$p_{2n}(0,0) = \binom{2n}{n} \frac{1}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$.

Stirling

$p_{odd}(0,0) = 0$.

$N! = \sqrt{2\pi N}\left(\frac{N}{e}\right)^N e^{\frac{\theta_N}{12N}}$

$(0 < \theta_N < 1)$.

d general case

$p_{odd}(0,0) = 0$.

$+\varepsilon_1 \cdots +\varepsilon_d \quad -\varepsilon_1 \cdots -\varepsilon_d$.

$p_{2n}(0,0) = \binom{2n}{n} \sum_{j_1 + \cdots + j_d = n} \frac{n!}{j_1! \cdots j_d!} \frac{n!}{j_1! \cdots j_d!} \times \frac{1}{(2d)^{2n}}$

the places of $\{+\varepsilon_1, \ldots +\varepsilon_d\}$

$+\varepsilon_1$ j_1 times

$\vdots$

$+\varepsilon_d$ j_d times.

$= \binom{2n}{n} \sum_{j_1 + \cdots + j_d = n} q_{j_1 \cdots j_d}^2$

where

$$q_{j_1\cdots j_d} := \frac{n!}{j_1!\cdots j_d!}\frac{1}{d^n}$$

$$\sum_{j_1\cdots j_d} q_{j_1\cdots j_d} = 1.$$

$d=2$.

$$q_{j_1 j_d} = \frac{n!}{j_1!\, j_2!}\frac{1}{2^n} = \begin{bmatrix} n \\ j_1 \end{bmatrix}\frac{1}{2^n}$$

$$j_1 = 0 \cdots n$$

$$p_{2n}(0,0) = \binom{2n}{n}\frac{1}{2^{2n}} \underbrace{\boxed{\sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}^2}}_{= \begin{bmatrix} 2n \\ n \end{bmatrix}} \frac{1}{2^{2n}}$$

$$= \binom{2n}{n}^2 \frac{1}{2^{4n}} = \left(\binom{2n}{n}\frac{1}{2^{2n}}\right)^2$$

$$\sim \frac{1}{\pi n}$$

$\rightarrow G(0,0) = +\infty$

$d \geq 3$.

$$p_{2n}(0.0) = \binom{2n}{n}\frac{1}{2^{2n}}\sum_{j_1\cdots j_d} q_{j_1\cdots j_d}^2$$

$$\leq \binom{2n}{n}\frac{1}{2^{2n}} \max_{j_1+\cdots+j_d=n} q_{j_1\cdots j_d}.$$

Let $\cancel{k_1}\, i_1, \dots, \cancel{k_d}\, i_d$ satisfies

$$q_{i_1 \cdots i_d} = \max_{j_1+\cdots+j_d=n} q_{j_1\cdots j_d}$$

Then $|i_k - i_l| \leq 1$.

Since if $i_k - i_l \geq 2$, then we would have

$$\frac{1}{i_k!\; i_l!} < \frac{1}{(i_k-1)!\,(i_l+1)!}.$$

$$q_{i_1\cdots i_d} \lneq q_{i_1 \cdots i_k - 1 \cdots i_l + 1 \cdots i_d}.$$

let $a := \min\{i_1 \cdots i_d\}$ $\times$ $(d-s)$ times

$a+1$ $\times$ (s) times

$\rightarrow n = a(d-s) + (a+1)s$

$= ad + s.$ $s = 0, \dots, d.$ $n \equiv s.$

Hence $\max = \dfrac{n!}{a!^{(d-s)}\,(a+1)!^{s}} \dfrac{1}{d^n}$

$n = ad + s \quad 0 \leq s \leq d$

$\max = \dfrac{n!}{a!^d} = \dfrac{n!}{(n/a)!^d} = \dfrac{n!}{(a+1)!^d} = \dfrac{n!}{(n/a)!^d}$

WMA $0 \leq s < d$.

$$p_{2n}(0,0) \leq \binom{2n}{n} \frac{1}{2^{2n}} \quad \frac{n!}{a!^{d-s}\,(a+1)!^{s}} \cdot \frac{1}{d^n}$$

$$\propto \frac{1}{\sqrt{\pi n}} \quad \frac{n!}{a!^d \quad (a+1)^s} \cdot \frac{1}{d^n}$$

$$\propto \frac{\left(\frac{n}{e}\right)^n}{(\sqrt{2\pi a})^d \left(\frac{a}{e}\right)^{ad} \quad (a+1)^s} \quad \frac{1}{d^n}$$

e^{-n+ad}
$=$
e^{-s}

$$\propto \frac{1}{\sqrt{a}^d} \quad \frac{n^n}{a^{n-s}} \quad \frac{1}{(a+1)^s} \cdot \frac{1}{d^n}$$

$$= \frac{1}{\sqrt{n}^d} \cdot \frac{n^n}{(n-s)^{n-s}} \cdot \frac{1}{(ad+d)^s}$$

$$\propto \frac{1}{n^{d/2}} \cdot \left(\frac{1}{1-s/n}\right)^n (n-s)^s \cdot \frac{1}{(n+d-s)^s}$$

$$\propto \frac{1}{n^{d/2}}.$$

Hence $d \geq 3 \rightarrow G(0,0) < \infty$.

<u>Proof 2</u> (Fourier analysis)

$$\widehat{\mathbb{Z}^d} = \mathbb{T}^d = [-\pi, \pi)^d.$$

$$t = (t_1 \cdots t_d) \in \mathbb{T}^d$$

$$\frac{dt}{\cancel{(2\pi)^d}} = \frac{dt_1 \cdots dt_d}{(2\pi)^d}$$

For $\mu \in \mathrm{Prob}(\mathbb{Z}^d)$.

$$\phi_\mu(t) := \sum_{x \in \mathbb{Z}^d} e^{-ix\cdot t}\mu(x)$$

inner prod

$$\rightsquigarrow \phi_{\mu * \nu} = \phi_\mu \phi_\nu$$

$$\phi_{\mu^{*n}} = \phi_\mu^n$$

inversion $\rightsquigarrow$

$$\mu^{*n}(x) = \int_{\mathbb{T}^d} \phi_\mu^n(t)\, e^{ix\cdot t}\, dt$$

If SRW, then

$$\phi_\mu(t) = \sum_{k=1}^{d} \frac{e^{-i\varepsilon_k\cdot t}}{2d} + \frac{e^{i\varepsilon_k \cdot t}}{2d} = \frac{\cos t_1 + \cdots + \cos t_d}{d}$$

Recall $p_\mu(x, y) = \mu(-x+y)$

$$(p_\mu)_n(x, y) = \mu^{*n}(-x+y) \leftarrow = 0 \text{ if } n \text{ odd}.$$

$\Downarrow$

$$G_\mu(x, y) = \sum_{n \geq 0} \mu^{*n}(-x+y)$$

$$= \sum_{n\geq 0} \int_{\mathbb{T}^d} \phi_\mu^n(t)\, e^{-i(-x+y)t}\, dt$$

$$G_\mu(0,0) = \sum_{n \geq 0} \int_{\mathbb{T}^d} \phi_\mu^n(t)\, dt$$

$$= \sum_{n\geq 0} \int_{\mathbb{T}^d} \phi_\mu^{2n}(t)\, dt$$

$$= \int_{\mathbb{T}^d} \frac{dt}{1 - \phi_\mu^2(t)}$$

$$< \infty.$$

Section 2 Harmonic functions.

§2.1 Harmonic & superharmonic functions.

(X, P) : irred. Markov chain
$(P\mathbb{1} = \mathbb{1})$

P acts on a function $f \in \mathbb{R}_+^X$

$$(Pf)(x) = \sum_{y \in X} p(x,y) f(y) \leq +\infty.$$

☆ For $f \in \ell^\infty(X)$ Pf well-defn

Defn. 2.1

$f : X \to \mathbb{R}_+$ is

(1) superharmonic if $Pf \leq f$

(2) harmonic if $Pf = f$

Notation.

$S^+(X, P) := \{ f \mid P\text{-superharm} \}$
or simply S^+

$H^+(X, P) := \{ f \mid P\text{-harmonic} \}$
or simply H^+

Rem. 2.2

- $\mathbb{R}_+ \mathbb{1} \subseteq H^+(X, P) \subseteq S^+(X, P) \subseteq \mathbb{R}_+^X$

  positive subcones.

- $H^+ \subset S^+$

  face in S^+

  Let $f_1, f_2 \in S^+$ & $0 < \lambda < 1$.

  If $\lambda f_1 + (1-\lambda) f_2 =: h \in H^+$,

  $$\begin{array}{ccccc} \lambda f_1 & + & (1-\lambda) f_2 & = & h \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \| \\ \lambda P f_1 & + & (1-\lambda) P f_2 & = & Ph \end{array}$$

  $\to Pf_1 = f_1 \quad Pf_2 = f_2$.

Lem 2.3 (Minimum principle)

Let $f \in S^+(X, P)$.
Suppose $\exists x_0 \in X$ s.t. $f(x_0) = \min\{f(x) \mid x \in X\}$.
Then $f(x) = f(x_0)$ $\forall x \in X$.

Proof.

$$f(x_0) \geq (Pf)(x_0) = \sum_y p(x_0, y) f(y) \geq \sum_y p(x_0, y) f(x_0) = f(x_0)$$

$\rightsquigarrow$ If $y \in X$, $p(x_0, y) > 0$ then $f(y) = f(x_0)$

By irreducibility, we are done. ▨

Lem 2.4 (Maximum principle)

Let $f \in H^+(X, P)$
Suppose $\exists x_0 \in X$. s.t. $f(x_0) = \max\{f(x) \mid x \in X\}$
Then $f(x) = f(x_0)$ $\forall x \in X$

Lem. 2.5 Stability

If $f_\lambda \in S^+(X, P)$, $\lambda \in \Lambda$, then $\bigwedge_{\lambda \in \Lambda} f_\lambda \in S^+(X, P)$

inf.

Proof.

$f := \inf_\lambda f_\lambda$

P preserves the order

$$\Rightarrow \quad Pf \leq Pf_\lambda \underset{f_\lambda \in S^+}{\leq} f_\lambda \quad \forall \lambda$$

▨

Lem. 2.6

$S^+(X, P)$ is closed in $\mathbb{R}_+^X$

Proof.

Let $\underset{S^+}{f_\lambda} \longrightarrow \underset{\mathbb{R}_+^X}{f}$ pt wise.

Fatou Check!

$$Pf(x) = \sum_y p(x, y) f(y) \leq \varliminf_\lambda \sum_y p(x, y) f_\lambda(y) \leq \varliminf_\lambda f_\lambda(x) = f(x)$$

▨

NOTE.
If P has fin. range (i.e. $\forall x$ $\#\{y \mid p(x,y) \neq 0\} < \infty$)
Then $H^+(X, P)$ is closed.
In general H^+ is Borel.

Thm. 2.7

(X, P) irr Markov chain

(1) If P recurrent, then $S^+(X,P) = \mathbb{R}_+ \mathbb{1}$

(2) If P transient, then $S^+(X,P) \supsetneq \mathbb{R}_+ \mathbb{1}$

Proof.

(1) Let $f \in S^+(X,P)$. & $k := f - Pf \geq 0$.

Then
$$\sum_{n=0}^{m} P^n k = f - P^{m+1} f \geq 0.$$

Since $P^m f \searrow$ (decreasing ptwise)

We have $h := \lim_{m\to\infty} P^m f$.

Thus $\sum_{n=0}^{\infty} P^n k(x)$ converges ptwise.

$$\begin{aligned} &\| \\ &\sum_{n=0}^{\infty} P_n(x,y)\, k(y) \quad \cancel{= Gk(x,y)} \\ &\| \\ &\sum_{n=0}^{\infty} G(x,y)\, k(y) \end{aligned}$$

Since $G(x,y) = +\infty \ \forall x, y$, this is possible only when $k \equiv 0$.

Hence $S^+(X,P) = H^+(X,P)$.

Take $f \in H^+(X,P)$. & Set

$$g := f \wedge f(x_0)\mathbb{1}.$$

Then $g \in S^+(X,P)$ by Lem
$$\begin{aligned} &\| \ \text{Now} \\ &H^+(X,P) \end{aligned}$$

But $g(x) \leq f(x_0) = g(x_0)$.

By maximal principle. $g \equiv f(x_0)$.
$\rightarrow f(x_1) \leq f(x)$

$\rightarrow f = \text{const.}$

(1) $\blacksquare$

(2) We show $G\delta_x \in S^+$. $\forall x \in X$.

$$PG\delta_x = (G - I)\delta_x \leq G\delta_x.$$

Suppose $G\delta_x = c\mathbb{1} \quad (c \in \mathbb{R}_+)$

$\downarrow I-P$

$\delta_x = c(I-P)\mathbb{1} = 0.$ ↯ (2) $\blacksquare$

<u>Ex</u>.2.8

$(\mathbb{Z}^d, P_\mu) \leftarrow$ SRW

$\to$ $d = 1, 2$ recurrent

$S^+ = H^+ = \mathbb{R}_+ \mathbb{1}$

$\to$ $d \geq 3$ transient.

$S^+ \supsetneq H^+ = \mathbb{R}_+ \mathbb{1}$

We will show this later.

§2.2 Potentials & Riesz decomposition

Defn. 2.9

(X, P) transient. (i.e. $G(x,y) < \infty$)

A function of the form Gk is called a potential with charge k

Rem. 2.10

- Charge is uniquely determined.

$$(I-P)\,Gk = k$$

- A potential is superharmonic

$$PGk = (G-I)k \leq Gk.$$

Notation.

$$\mathcal{P}^+(X,P) := \{ Gk \mid k \geq 0 \}$$

~~We~~ s.t. $Gk(x) < \infty$.

$\to \mathcal{P}^+ \subset S^+$ subcone.

Thm. 2.11 (Riesz decomposition)

(X, P) transient. Then.

$$\left(S^+(X,P) = \mathcal{P}^+(X,P) + \mathcal{H}^+(X,P) \right)$$

unique dec.

$\forall f \in S^+$ $\exists!\, g \in \mathcal{P}^+$, $\exists!\, h \in \mathcal{H}^+$ s.t. $f = g + h$

Proof

$f \geq Pf \geq P^2 f \geq \cdots$ ptwise decreasing

$\to$ $h(x) := \lim_{n\to\infty} (P^n f)(x)$ well-defined.

$$= \lim_n P P^{n-1} f(x) \underset{\text{Fatou}}{\geq} Ph(x)$$

$$(Ph)(x) = \sum_y p(x,y)\, h(y) = \sum_y p(x,y) \lim_n \underbrace{P^n f(y)}_{\leq f(y)}$$

$$\overset{\text{Lebesgue}}{=} \lim_n \sum_y p(x,y)\, P^n f(y) = h(x)$$

Next let $k := f - Pf \geq 0$

$$\sum_{n=0}^{m} P^n k = f - P^{m+1} f \nearrow f - h$$

$$\uparrow_{m\to\infty}$$

$$\sum_{n=0}^{\infty} P^n k = Gk. \quad \rightsquigarrow \quad Gk = f - h$$

∎

<u>Uniqueness</u>

$f = \cancel{G} k' + h'$ another dec.

$Pf = PGk' + h'$

$\downarrow$

$(I-P)f = (G - PG)k' = k'$.

$\|$

k

∎

Lem. 2.12

(X, P) transient

(1) $f \in \cancel{\mathbb{R}_+^X}\, S^+$ $\quad f \in \mathcal{P}^+ \iff \cancel{f \in S^+}\; P^n f \searrow 0$.

(2) $\mathcal{P}^+(X, P) \subseteq S^+(X, P)$ is hereditary.

i.e. if $\underset{S^+}{\underset{\cap}{0 \leq f}} \leq \underset{\mathcal{P}^+}{\underset{\cap}{g}}$ then $f \in \mathcal{P}^+$

(3) $\mathcal{P}^+(X, P) \subseteq S^+(X, P)$ face

<u>Proof</u>

(1) Trivial from Riesz dec.

(2) Trivial from (1).

(3) $f_1, f_2 \in S^+$ & $0<\lambda<1$.

If $\lambda f_1 + (1-\lambda) f_2 \in \mathcal{P}^+$

By hereditarity $f_1, f_2 \in \mathcal{P}^+$.

∎

☆ From (1) it turns out that $\mathcal{P}^+ \subset \mathbb{R}_+^X$ is Borel.

§2.3 Extremal superharmonics.

We fix $o \in X$. We define convex sets:

$$S_o^+(X, P) := \{ f \in S^+(X, P) \mid f(o) = 1 \}$$

$$\mathcal{P}_o^+(X, P) := \{ f \in \mathcal{P}^+(X, P) \mid f(o) = 1 \}$$

$$H_o^+(X, P) := \{ f \in H^+(X, P) \mid f(o) = 1 \}$$

☆ $\mathcal{P}_o^+ \underset{\text{face}}{\subset} S_o^+ \quad H_o^+ \underset{\text{face}}{\subset} S_o^+$.

Lem. 2.13

$$\boxed{S_o^+(X, P) \text{ is a compact convex set in } \mathbb{R}_+^X}$$

Proof.

We know $S_o^+(X, P)$ is closed.

Let $x \in X$. By irreducibility,

$\exists n_x \in \mathbb{N}$ s.t. $p_{n_x}(o, x) > 0$.

Then for $f \in S_o^+$, we have

$$1 = f(o) \geq (P^{n_x} f)(o) = \sum p_{n_x}(o, y) f(y)$$
$$\geq p_{n_x}(o, x) f(x).$$

$$\to \quad 0 \leq f(x) \leq p_{n_x}(o, x)^{-1}.$$

Hence.
$$S_o^+ \subset \prod_{x \in X} [0, p_{n_x}(o, x)^{-1}]$$

▨

☆ Krein–Milman thm says.

$$S_o^+ = \overline{\mathrm{co}}^w(\mathrm{ex}(S_o^+))$$

→ Want to study $\mathrm{ex}(S_o^+)$.

☆ Choquet theory

$$f(x) = \int_{\mathrm{ex}(S_o^+)} g(x) \, \mu(dg)$$

$$\mathrm{ex}(S_o^+) \leftrightarrow X \cup \partial_M^{\min} X.$$

Lem. 2.14

$$\boxed{ex(S_0^+) = ex(\mathcal{P}_0^+) \sqcup ex(\mathcal{H}_0^+)}$$

Proof.

$\supset$: Because $\mathcal{P}_0^+, \mathcal{H}_0^+ \subset S_0^+$ faces.

$\subset$: Let $f \in ex(S_0^+)$

$$\underset{}{f} = \underset{\mathcal{P}^+}{\underset{\cap}{g}} + \underset{\mathcal{H}^+}{\underset{\cap}{h}} \quad \text{(Riesz dec)}$$

$$= g(0) \underbrace{\frac{g}{g(0)}}_{\mathcal{P}_0^+} + h(0) \underbrace{\frac{h}{h(0)}}_{\mathcal{H}_0^+}$$

extremal $\rightarrow$ $f = \dfrac{g}{g(0)} \in \mathcal{P}_0^+$, & $f = \dfrac{h}{h(0)} \in \mathcal{H}_0^+$

$\cancel{\& \ h = 0}$ $\rightarrow$ $\lightning$

$\rightarrow$ $f = g$ or h ∎

☆ $h \in \mathcal{H}^+$ is <u>minimal</u> when

if ~~h is a minimal element of~~ $\cancel{\mathcal{H}^+}$ ~~&~~

$\underset{\mathcal{H}^+}{\underset{\cap}{h'}} \leq h$, then $\exists c \geq 0$ $h' = ch$.

Easy to see that for $h \in \mathcal{H}_0^+$,

h $\cancel{\in ex(\mathcal{H}_0^+)}$ extremal $\Leftrightarrow$ h is minimal

Lem. 2.15

$$\boxed{ex(\mathcal{P}_0^+) = \left\{ \frac{G(\cdot,y)}{G(0,y)} \;\middle|\; y \in X \right\}}$$

Proof.

NOTE $\dfrac{G(\cdot,y)}{G(0,y)} = \dfrac{G\delta_y}{G(0,y)} \in \mathcal{P}_0^+$.

$\subset$: Let $g = Gk \in ex(\mathcal{P}_0^+)$

$$g(x) = \sum_{y \in X} G(x,y)\, k(y)$$

$$1 = g(0) = \sum_{y \in X} G(0,y)\, k(y)$$

prob measure on X.

$$g(x) = \sum_{y \in X} \underset{\mathcal{P}_0^+}{\underset{\cap}{\frac{G(x,y)}{G(0,y)}}}\, G(0,y)\, k(y)$$

extremality $\exists y \in X \quad g(x) = \dfrac{G(x,y)}{G(0,y)} \quad \forall x \in X$.

$\supset$: Suppose for $0 < \lambda < 1$, $k_1, k_2 \geqq 0$

$$\frac{G\delta_y}{G(0,y)} = \lambda G k_1 + (1-\lambda) G k_2$$

$\downarrow (1-P)\times$

$$\frac{\delta_y}{G(0,y)} = \lambda k_1 + (1-\lambda) k_2$$

$\rightsquigarrow$ $k_1 = c_1 \delta_y$, $k_2 = c_2 \delta_y$

$\mathcal{P}_0^+ \ni G k_1 = c_1 G \delta_y$

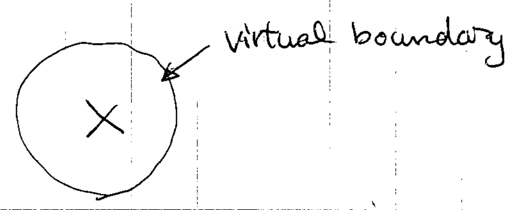
$\to$ $c_1 = G(0,y)^{-1} = c_2$

$\to$ $k_1 = k_2 = G(0,y)^{-1} \delta_y$

☆ Hence $ex(\mathcal{P}_0^+) \cong X$ as a set.

$$\begin{array}{ccc} ex(\mathcal{P}_0^+) & \cong & X \\ \Psi & & \Psi \\ \dfrac{G\delta_y}{G(0,y)} & \longleftrightarrow & y \end{array}$$

How about $ex(\mathcal{H}_0^+)$?



$ex(\mathcal{H}_0^+) \longleftrightarrow \partial_M^{min} X$.

☆ $\mathrm{ex}\, H_0^+ \neq \emptyset$

NOTE $\underset{\text{Borel}}{\underset{\uparrow}{H_0^+}} \subset \underset{\text{cpct convex}}{\underset{\uparrow}{S_0^+}}$

$H^\infty(X, P) := \{ f \in \ell^\infty(X) \mid Pf = f \}$

$\rightsquigarrow$ $H^\infty \subset \ell^\infty(X)$ $\underset{\uparrow}{\sigma\text{-weakly}}$ closed OS.

$P \curvearrowright \ell^\infty(\overset{X}{\cancel{\mathbb{Z}}})$ σ-w cont..

→ H^∞'s closed unit ~~ball~~ σ-w cpct.

$\mathrm{ex}\, H^\infty$ ball $\neq \emptyset$

$\mathrm{ex}\, H^\infty_+$ ball $\neq \emptyset$.

$\underset{\text{face}}{\underset{\uparrow}{H^\infty \cap H_0^+}} \subset H_0^+$

§2.4 Choquet–Deny theory

Γ : discrete abelian grp, denumerable

0 : the neutral element.

$\mu \in \mathrm{Prob}(\Gamma)$ non-deg.

$P_\mu = (p_\mu(x,y))_{x,y}$, $p_\mu(x,y) = \mu(-x+y)$

We will describe $ex\, H_0^+$.

$f \geq 0$.

$$(P_\mu f)(x) = \sum_y p_\mu(x,y) f(y)$$
$$= \sum_y \mu(-x+y) f(y)$$
$$= \sum_y \mu(\underline{\underline{y}}) f(x+\underbrace{\underline{\underline{y}}}_{\text{(right) transl.}})$$

P_μ commutes with transls

$\rightarrow P_\mu H^+ \subseteq H^+$

If $h \in ex(H_0^+)$, then

$$h(x) = \sum_y \mu(y) \underbrace{h(x+y)}_{H^+ \text{ w.r.t. } x}$$

$$= \sum_y \underbrace{\mu(y) h(y)}_{\text{new prob measure}} \underbrace{\frac{h(x+y)}{h(y)}}_{H_0^+ \text{ wrt } x}.$$

$1 = h(0) = \sum_y \mu(y) h(y) = \cancel{\ }$.

extremality $\rightsquigarrow$ $h(x) = \dfrac{h(x+y)}{h(y)}$ $\forall y \in \mathrm{supp}\,\mu$ $\forall x \in \Gamma$.

non-deg $\rightsquigarrow$ $h(x+y) = h(x) h(y)$ $\forall x, y \in \Gamma$.

Let

$$\mathcal{E}_\mu := \{ f \mid f : \Gamma \to \mathbb{R}_+^* \text{ "grp homo" "exponential"} \ \& \ \sum_{y \in \Gamma} \mu(y) f(y) = 1 \}$$

Then

$$ex(H_0^+) \subset \mathcal{E}_\mu.$$

☆ We claim that

$$H^\infty(\Gamma, P_\mu) = \mathbb{C}1. \quad \forall \mu \in \mathrm{Prob}(\Gamma) \text{ non-deg}.$$

Indeed.

$$\mathrm{ex}(H^\infty_+ \cap \underset{H_0^+}{\cancel{\mathrm{Ball}}}) \subset \mathrm{ex}\, H_0^+ \subset \underset{\substack{= \{1\} \\ \text{or} \\ \text{contains} \\ \text{unbdd. exp.}}}{\mathcal{E}_\mu}$$

Thm 2.16 (Choquet–Deny)

$$\boxed{\mathrm{ex}\,(H_0^+(\Gamma, P_\mu)) = \mathcal{E}_\mu}$$

Proof.

$\subset$ ok

$\supset$ Let $h \in \mathcal{E}_\mu$.

We will check the minimality.

Let $f \in H^+$ & $f \leq h$.

Consider $0 \leq \frac{f}{h} \leq 1$. bdd.

$$P_\mu^h := (p_\mu^h(x,y))_{x,y}.$$

$$p_\mu^h(x,y) = \frac{1}{h(x)} p_\mu(x,y)\, h(y).$$

$$\sum_y p_\mu^h(x,y) \underset{\text{harmonicity}}{=} \frac{1}{h(x)} h(x) = 1$$

$\to P_\mu^h$ new Markov chain

$$\left(P_\mu^h \cdot \frac{f}{h}\right)(x) = \sum_y p_\mu^h(x,y) \frac{f(y)}{h(y)}$$

$$= \sum_y \frac{1}{h(x)} p_\mu(x,y) f(y)$$

$$= \frac{f(x)}{h(x)}.$$

$\to \frac{f}{h}$ is P_μ^h – harmonic.

$$\to \frac{f}{h} \in H^\infty(\Gamma, P_\mu^h)_+ \underset{\text{exp}}{=} H^\infty(\Gamma, P_{\mu h})_+ = \mathbb{R}_+ 1$$

$$P_{\mu h}(x,y) = \mu h(-x+y) = \mu(-x+y)\, h(-x)\, h(y)$$

This technique is called the <u>h-process.</u>

Let (X, P) Markov & $h \in H^+$ non-zero.

$$P^h := (p^h(x, y))_{x, y \in X}$$

$$p^h(x, y) = \frac{1}{h(x)} p(x, y) h(y)$$

(X, P^h) new Markov chain.

Easy to see that

$$S^+(X, P) \longrightarrow S^+(X, P^h)$$

bijective, affine.

$$H^+(X, P) \longrightarrow H^+(X, P^h)$$

$$f \longmapsto f/h$$

If $h(o) = 1$, this map induces

$$S_o^+(X, P) \longrightarrow S_o^+(X, P^h)$$

$$H_o^+(X, P) \longrightarrow H_o^+(X, P^h)$$

& the bijection between the extremals.

<u>NOTE.</u>

$$(P^h)^n =: (p_n^h(x, y))_{x, y}$$

$$p_n^h(x, y) = \sum_{x_1 \cdots x_{n-1}} p^h(x, x_1) \cdots p^h(x_{n-1}, y)$$

$$= \frac{1}{h(x)} p_n(x, y) h(y).$$

$$G^h(x, y) = \frac{1}{h(x)} G(x, y) h(y).$$

☆ $$\frac{G^h(x, y)}{G^h(o, y)} = \frac{1}{h(x)} \frac{G(x, y)}{G(o, y)}$$

☆ If $h \in ex(H_o^+)$,

$$H^\infty(X, P^h)_+ = \mathbb{R}_+ \mathbb{1}.$$

$$\Updownarrow$$

$$H^+(X, P) \cap \{f \mid f \leq ch\} = \mathbb{R}_+ h.$$

Section 3. Martin boundary

§3.1 Martin compactification.

(X, P) : irreducible & transient. $G(x,y)<\infty \quad \forall x,y \in X$.

$o \in X$: fixed origin.

Defn. 3.1

Set $K(x,y) := \dfrac{G(x,y)}{G(o,y)} \quad x,y \in X.$

$$K := (K(x,y))_{x,y\in X}$$

We call $K(x,y)$ the Martin kernel

Rem. 3.2

- $\forall x \in X \ \exists C_x, C_x' > 0$ s.t.

  $C_x < K(x,y) < C_x' \quad \forall y \in X.$

  bdd w.r.t. y

- $\operatorname{ex} \mathcal{P}_o^+ = \{ K(\cdot, y) \mid y \in X \} = \{ K\delta_y \mid y \in X \}$

Defn. 3.3

$\hat{X}_M :=$ the compactification of X so that $X \ni y \longmapsto K(x,y) \in \mathbb{R}_+$ continuously extends to $\hat{X}_M$. $(\forall x \in X)$

We call $\hat{X}_M$ the Martin compactification

& $\partial_M X := \hat{X}_M \setminus X$ the Martin boundary

Rem. 3.4

- $C(\hat{X}_M) = C_0(X) + C^*(K(x,\cdot) \mid x \in X)$.

  $\subseteq \ell^\infty(X)$

  $\rightarrow X \subset \hat{X}_M$ (open) dense.

  every $x \in X$ is open.

  $\partial_M X$ cpct.

- $C(\partial_M X) \cong C(\hat{X}_M)/C_0(X)$

  $f\restriction_{\partial_M X} \longleftarrow\!\shortmid f + C_0(X)$

- $\hat{X}_M$ doesn't dep on o. since $K_{o'}(x,y) = \dfrac{G(o,y)}{G(o',y)} K(x,y) = \dfrac{K(x,y)}{K(o',y)}$

Notation

- $K(x,\cdot) \in C(\hat{X}_M)$.

  For $\alpha \in \hat{X}_M$, $K(x,\alpha) := \langle K(x,\cdot), \alpha \rangle$

  $\hat{X}_M$: Gelfand spectrum.

Rem. 3.5

- $\hat{X}_M \ni \alpha_n \longrightarrow \alpha \in \hat{X}_M \overset{\text{by defn}}{\Longleftrightarrow} \langle f, \alpha_n \rangle \longrightarrow \langle f, \alpha \rangle$ $\forall f \in C_0(X) + \mathbb{C}(\ )$

  if $\alpha \in X$, $\exists N \geq 1$ s.t. $n \geq N \Rightarrow \alpha_n = \alpha \in X$

  if $\alpha \in \partial_M X$, $\alpha_n \to \alpha \Longleftrightarrow K(x,\alpha_n) \to K(x,\alpha)$ $\forall x$.

  $f(\alpha_n) \to 0$ $f \in C_0(X)$

- $K(\cdot,\alpha) \in S_0^+(X,P)$. $\forall \alpha \in \hat{X}_M$.

  Indeed, $X \ni y_n \longrightarrow \alpha$, then

  $\mathcal{P}_0^+ \ni K(\cdot, y_n) \longrightarrow K(\cdot,\alpha)$ ptwise

- When P is of finite range,

  (i.e. $\#\{y \mid p(x,y)\} < \infty$ $\forall x \in X$)

  $K(\cdot,\alpha) \in H_0^+$ $\forall \alpha \in \partial_M X$.

  Let $y_n \to \alpha$ (1st var.)

$$(PK(\cdot, y_n))(x) = \sum_{y \text{ fin sum}} p(x,y) K(y,y_n)$$
$$= \sum_y p(x,y) \frac{G(y,y_n)}{G(0,y_n)}$$
$$= \frac{G(x,y_n) - \delta_{x,y_n}}{G(0,y_n)}$$
$$= K(x,y_n) - \frac{\delta_{x,y_n}}{G(0,y_n)}$$

Letting $n \to \infty$, we obtain

$$\sum_{y \text{ fin sum}} p(x,y) K(y,\alpha) = K(x,\alpha).$$

since $\delta_{x,y_n} = 0$ if $n >> 1$.

Thm. 3.6 (Poisson – Martin integral repn)

$$\boxed{\begin{array}{l} {}^{\forall} h \in H^{+}(X, P) \\ {}^{\exists}\mu : \text{Radon measure on } \partial_M X. \\ \text{s.t.} \\ \qquad h(x) = \displaystyle\int_{\partial_M X} K(x, \alpha)\, \mu(d\alpha) \end{array}}$$

Rem. 3.7

μ is not unique in general.

$\longrightarrow$ minimal boundary

Lem. 3.8

$$\boxed{{}^{\forall} h \in H^{+} \quad {}^{\exists} g_n \in \mathcal{P}^{+}, n \in \mathbb{N} \quad g_n \nearrow h}$$

Proof. (Balayage Technique)

$A \subset\subset X$ fixed.

$g_A := \inf \{ f \in S^{+} \mid h \leq f \text{ on } A \}$

$\rightsquigarrow \cdot\ g_A \in S^{+}$ (Lem 2.5)

$\cdot\ g_A \leq h$. ($\{\ \} \ni h$)

$\cdot\ g_A = h$ on A ($h \leq g_A$ on A).

We show $g_A \in \mathcal{P}^{+}$, then we are done.

(NOTE: $A \subset B \subset\subset X \Rightarrow g_A \leq g_B$).

$$G(h 1_A)(x) = \sum_{y \in X} G(x, y)\, (h 1_A)(y)$$

$$= \sum_{y \in A} G(x, y)\, h(y)$$

$$\overset{\text{if } x \in A}{\geq} G(x, x)\, h(x)$$

$$\geq h(x).$$

$\rightarrow \quad g_A \leq \underset{\mathcal{P}^{+}}{\underset{\cap}{G(h 1_A)}}$

$\rightarrow \quad g_A \in \mathcal{P}^{+}$ (Lem 2.12). ▨

~~Lem~~ Proof of Thm 3.6

Let $h \in H^+$ take $\underset{\mathcal{P}^+}{\underset{\cap}{g_n}} \nearrow h$.

$g_n = G f_n \quad f_n \geq 0$.

$$g_n(x) = \sum_{y \in X} G(x,y) f_n(y)$$

$$= \sum_{y \in X} K(x,y)\, G(0,y) f_n(y)$$

$$= \int_{\hat{X}_M} K(x,\alpha)\, \mu_n(d\alpha)$$

$$\mu_n = \sum_{y \in X} G(0,y) f_n(y)\, \delta_y$$

$$\to \mu_n(\hat{X}_M) = \sum_{y \in X} G(0,y) f_n(y)$$

$$= G f_n(0) \leq h(0).$$

unif. bdd.

Take a subseq $\mu_{n_k} \xrightarrow{w^*} \mu$. in $M(\hat{X}_M)$.

Then $h(x) = \int_{\hat{X}_M} K(x,\alpha)\, \mu(d\alpha) \quad \forall x \in X$.

We show $\mu(\{y\}) = 0 \quad \forall y \in X$.

$$\underset{H^+}{\underset{\cap}{h(x)}} = \underset{\mathcal{P}^+}{\underset{\cap}{K(x,y)\mu(\{y\})}} + \underset{S^+}{\underset{\cap}{\int_{\hat{X}_M \setminus \{y\}} K(x,\alpha)\, \mu(d\alpha)}}$$

By Riesz decomposition, (Thm 2.11) $K(x,y)\mu(\{y\}) = 0$

$\blacksquare$

§3.2 Minimal Martin Boundary

Each extremal (minimal) harmonic has unique integral repn.

Prop. 3.9

Let $h \in ex(H_0^+)$ &

$$h(x) = \int_{\partial_M X} K(x,\alpha)\,\mu(d\alpha)$$

an integral repn of h w.r.t. $\mu \in \mathrm{Prob}(\partial_M X)$

Then $\exists !\ \alpha \in \partial_M X$ s.t. $\mu = \delta_\alpha$

Proof.

Let $B \subset \partial_M X$ Borel.

$$\underset{\substack{\cap \\ ex(H_0^+) \\ \cap \\ H_0^+ \underset{\text{face}}{\subset} S_0^+}}{h(x)} = \underset{\substack{\cap \\ S^+}}{\int_B K(x,\alpha)\,\mu(d\alpha)} + \underset{\substack{\cap \\ S^+}}{\int_{\partial_M X \setminus B} K(x,\alpha)\,\mu(d\alpha)}$$

$\rightsquigarrow$ extremality (or minimality)

$$\exists \lambda_B \geq 0 \quad \lambda_B h(x) = \int_B K(x,\alpha)\,\mu(d\alpha)$$

Put $x = 0$ (origin), then

$$\lambda_B = \mu(B).$$

Thus

$$\int_B h(x)\,\mu(d\alpha) = \int_B K(x,\alpha)\,\mu(d\alpha)$$

$\forall B \subset \partial_M X$ Borel.

$\rightsquigarrow \forall x \in X.\quad h(x) = K(x,\alpha) \quad \mu\text{-a.e. }\alpha.$

$\rightsquigarrow$ (X denumerable) μ-a.e. α. $\quad h(x) = K(x,\alpha) \quad \forall x \in X$.

Since $\partial_M X \ni \alpha \longmapsto K(x,\alpha)$ is injective.

Such α is unique. & $\mu = \delta_\alpha$ ▨

We have shown

$$\mathrm{ex}\, H_o^+ \subset \{ K(\cdot,\alpha) \mid \alpha \in \partial_M X \}.$$

Defn 3.10 (Minimal Martin boundary)

$$\boxed{\partial_M^{\min} X := \{ \alpha \in \partial_M X \mid K(\cdot,\alpha) \in \mathrm{ex}\, H_o^+ \}}$$

☆ $\underset{\substack{\uparrow \\ \text{Borel}}}{\partial_M^{\min} X} \subset \partial_M X$

Since

$$\begin{array}{ccc} \partial_M X & \xrightarrow{\pi} & S_o^+ \\ \cup\!\!\!\!\! & & \cup\!\!\!\!\! \\ \alpha & \longmapsto & K(\cdot,\alpha) \end{array} \quad \text{continuous injective.}$$

$$\partial_M^{\min} X = \pi^{-1}(\underset{\substack{\uparrow \\ \text{Borel.}}}{\mathrm{ex}\, H_o^+}).$$

☆ In many examples, $\partial_M^{\min} X = \partial_M X$.

Thm. 3.11 (Unique integral repn)

(X, P) : Irr & translnt.

$\forall h \in H^+(X,P) \quad \exists!\, \mu_h \in M(\partial_M^{\min} X)$

s.t.

$$h(x) = \int_{\partial_M^{\min} X} K(x,\alpha)\, \mu_h(d\alpha) \qquad \forall x \in X.$$

This is the main result of this lecture.

We present a proof using RW.

☆ We discuss the stability of $\hat{X}_M$ wrt the h-process.

Let (X, P) as before & $h \in H^+$.

$$P^h = (p^h(x,y))_{x,y}$$

$$p^h(x,y) = \frac{1}{h(x)} p(x,y) h(y).$$

$$\to G^h(x,y) = \sum_{n \geq 0} p^h_n(x,y) = \frac{1}{h(x)} G(x,y) h(y)$$

$$\to K^h(x,y) = \frac{G^h(x,y)}{G^h(0,y)} = \frac{h(0)}{h(x)} K(x,y)$$

$$\to C^*(K^h(x,\cdot))_x = C^*(K(x,\cdot))_x$$

Thus $\hat{X}^h_M = \hat{X}_M$.

Moreover,

$$\begin{array}{ccc} \mathrm{ex}\, H^+_0(X, P^h) & \overset{\mathrm{bij}}{\longleftrightarrow} & \mathrm{ex}\, H^+_0(X, P) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ h(0)\frac{f}{h} & \longleftrightarrow & f \end{array}$$

Hence

$\alpha \in \partial_M X$ satisfies $K(\cdot, \alpha) \in \mathrm{ex}\, H^+_0(X, P)$

$$\Longleftrightarrow \underbrace{\frac{h(0)}{h} K(\cdot, \alpha)}_{= K^h(\cdot, \alpha)} \in \mathrm{ex}\, H^+_0(X, P^h)$$

i.e. $\partial^{\min}_M X^h = \partial^{\min}_M X$.

☆ To prove Thm 3.11, it suffices to show.

$\exists! \ \mu \in M(\partial_M^{min} X)$ s.t

$$1 = \int_{\partial_M^{min} X} K(x,\alpha)\, \mu_1(d\alpha)$$

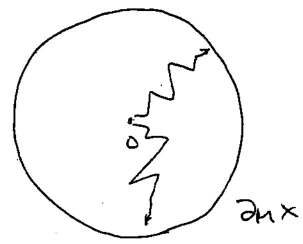
Indeed, if $h \in H^+$ we consider the h-process (X, P^h), & apply

$$1 = \int_{\partial_M^{min} X} K^h(x,\alpha)\, \mu_1^h(d\alpha)$$

$$= \int_{\partial_M^{min} X} \frac{h(0)}{h(x)} K(x,\alpha)\, \mu_1^h(d\alpha)$$

$$\rightarrow \ h(x) = \int_{\partial_M^{min} X} K(x,\alpha)\, h(0)\, \mu_1^h(d\alpha).$$

☆ Strategy



We show a RW converges to $\partial_M^{(min)} X$ almost surely.

Then push the prob measure $\mathbb{P}_0$ to $\partial_M^{(min)} X$.

i.e. the hitting probability.

Section 4. Convergence to the Boundary.

§4.1 Path space

(X, P) : Markov chain $P\mathbb{1} = \mathbb{1}$.
(irred. not assumed)

$\Omega := X^{\mathbb{N}}$, $\mathbb{N} = \{0, 1, 2, \cdots\}$

$$\begin{array}{cccc} Z_k : & \Omega & \longrightarrow & X \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (\omega_n) & \longmapsto & \omega_k \end{array}$$ the k-th coordinate

$\mathcal{F}_n :=$ the σ-field gen. by

$Z_k^{-1}(A_k)$ $\quad A_k \in 2^X$

$k = 0, \cdots, n$.

$=$ the σ-field with atoms

$C_{a_0 \cdots a_n} := \{a_0\} \times \cdots \times \{a_n\} \times X \times \cdots$

the cylinder sets

Then $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$

Set $\mathcal{F} := \sigma\left(\bigcup_{n \geq 0} \mathcal{F}_n\right)$

Thm 4.1 (Kolmogorov extension thm)

$\forall \gamma$: measure on X $(\gamma(pt) < \infty.)$

$\exists!$ $\mathbb{P}_\gamma$: σ-fin. measure on $(\Omega, \mathcal{F})$

s.t.

$\mathbb{P}_\gamma(C_{a_0 \cdots a_n}) = \gamma(a_0)\, p(a_0, a_1) \cdots p(a_{n-1}, a_n)$

$\forall a_0, \ldots, a_n \in X$

Rem. 4.2.

- $\mathbb{P}_\gamma(\Omega) = \sum_{a \in X} \gamma(a)$. Hence if $\gamma \in \mathrm{Prob}(X)$, $\mathbb{P}_\gamma \in \mathrm{Prob}(\Omega)$. γ is called an <u>initial distr.</u>
- For $\gamma = \delta_x$, $\mathbb{P}_x := \mathbb{P}_{\delta_x}$.
- $\mathbb{P}_\gamma = \sum_x \gamma(x)\, \mathbb{P}_x$
- $\mathbb{P}_x(Z_n = y) = p_n(x, y)$

Thm. 4.3 (Convergence to the Boundary)

(X, P) : irred & transient.

Then $\forall \gamma \in \mathrm{Prob}(X)$ initial distr.

$$\underset{X}{\underset{\cap}{Z_n(\omega)}} \xrightarrow[n\to\infty]{} {}^{\exists}\underset{Z_\infty(\omega)}{\text{limit}} \in \partial_M X \quad \mathbb{P}_\gamma\text{-a.s.}\,\omega$$

We prove this result later.

Lem. 4.4

Let $B \in \mathcal{F}$. $(B \subset \Omega)$ & set

$$f(x) := \mathbb{P}_x(B) \geq 0, \quad x \in X.$$

Then

$$\underset{\substack{\uparrow \\ \text{trans. matrix}}}{(P} f)(x) = \mathbb{P}_x(\sigma^{-1}(B)), \quad {}^{\forall} x \in X$$

where $\sigma : \Omega \to \Omega \quad (\omega_0, \cdots) \mapsto (\omega_1 \cdots)$

the time shift

Proof.

It suffices to show it for $B = C_{a_0 \cdots a_n}$.

$$(Pf)(x) = \sum_y p(x, y) f(y)$$

$$= \sum_y p(x, y) \mathbb{P}_y(C_{a_0 \cdots a_n})$$

$$= \sum_y p(x, y) \delta_y(a_0) p(a_0, a_1) \cdots p(a_{n-1}, a_n)$$

$$= p(x, a_0) p(a_0, a_1) \cdots p(a_{n-1}, a_n).$$

$$\sigma^{-1}(B) \ni \omega \iff \sigma(\omega) \in C_{a_0 \cdots a_n}$$

$$\iff \omega_1 = a_0, \quad \cdots \quad \omega_{n+1} = a_n$$

$$\iff \omega \in X \times \{a_0\} \times \cdots \times \{a_n\} \times X \times \cdots$$

$$\mathbb{P}_x(\sigma^{-1}(B)) = \mathbb{P}_x\Big(\sum_{y \in X} C_{y, a_0 \cdots a_n}\Big)$$

$$= \sum_y \delta_x(y) p(y, a_0) \cdots p(a_{n-1}, a_n)$$

$$= p(x, a_0) \cdots p(a_{n-1}, a_n)$$

§4.2 Recurrence & Transience

Let $A \subset\subset X$. Set

$$u_A(x) := \mathbb{P}_x\left(\{\omega \in \Omega \mid Z_n(\omega) \in A \text{ for some } \underline{\underline{n \geq 0}}\}\right)$$

$$\overset{\text{simply}}{=} \mathbb{P}_x(Z_n \in A \text{ for some } \underline{\underline{n \geq 0}})$$

$\downarrow$ time shift

$\xrightarrow{\text{Lem 4.4}}$ $P u_A(x) = \mathbb{P}_x(Z_n \in A \text{ for some } n \geq 1)$

$\vdots$

$$P^k u_A(x) = \mathbb{P}_x(Z_n \in A \text{ for some } n \geq k)$$

In particular, $u_A \geq P u_A$ i.e. $u_A \in S^+$.

$\left(\begin{array}{l}\text{In general, if } B \in \mathcal{F} \text{ satisfies } \sigma^{-1}(B) \subset B \\ \text{then } x \mapsto \mathbb{P}_x(B) \in S^+\end{array}\right)$.

Letting $k \to +\infty$

$$\lim_{k\to\infty} (P^k u_A)(x) = \mathbb{P}_x(Z_n \in A \ \infty \text{ many } n\text{'s})$$

Thm. 4.5

(X, P) irred Markov.

(1) If P recurrent, then $\forall x \in X$ $\forall A \subset\subset X$

$$\mathbb{P}_x(Z_n \in A \ \infty \text{ many } n\text{'s}) = 1$$

(2) If P transient, then $\forall x \in X$ $\forall A \subset\subset X$

$$\mathbb{P}_x(Z_n \in A \ \infty \text{ many } n\text{'s}) = 0$$

Proof.

(1) By Thm 2.7, $u_A \in S^+ = \mathbb{R}_+ \mathbb{1}$.

Hence $u_A(x) = u_A(\underset{\substack{\Uparrow \\ A}}{a}) = 1$.

$\rightarrow \lim_{k\to\infty} P^k u_A = \mathbb{1}$

(2) We will show $u_A \in \mathcal{P}^+$.

Then from Lem , we see $\lim_{k\to\infty} P^k u_A = 0$.

$$u_A(x) = \mathbb{P}_x\left(\bigcup_{n\geq 0} \{z_n \in A\}\right)$$

$$\leq \sum_{n\geq 0} \mathbb{P}_x(z_n \in A)$$

$$= \sum_{n\geq 0}\sum_{y\in A} \mathbb{P}_x(z_n = y)$$

$$= \sum_{n\geq 0}\sum_{y\in A} p_n(x.y)$$

$$= \sum_{y\in A} G(x.y) \quad (<\infty).$$

$$\to u_A \leq \sum_{y\in A} G\,\delta_y$$

Since $u_A \in S^+$, $u_A \in \mathcal{P}^+$ by Lem 2.12(2)

∎

Hence inf P transient, $\exists k \geq 0$ s.t. ℓ_A charge

$$G\ell_A(x) = \mathbb{P}_x(z_n \in A \text{ for some } n\geq 0)$$

$\downarrow (I-P)\times$

$$\ell_A(x) = \mathbb{P}_x(z_n \in A \text{ for some } n \geq 0)$$

$$- \mathbb{P}_x(z_n \in A \text{ for some } n \geq 1)$$

$$= \mathbb{P}_x(z_0 \in A \ \&\ z_n \notin A\ n \geq 1)$$

In particular, if $A \ni 0$, then

$$G\ell_A(0) = 1$$

In general

$$\sum_y G(x,y)\,\ell_A(y) = \mathbb{P}_x(z_n \in A \text{ for some } n\geq 0)$$

$$\leq 1.$$

One more comments

If $A_n \subset\subset X$, $A_n \nearrow X$

then $u_{A_n}(x) = \mathbb{P}_x(Z_m \in A_n \text{ for some } m \geq 0)$

$\nearrow 1 \quad (n \to \infty)$

i.e.

$$\begin{array}{ccc} u_{A_n} & \nearrow & \mathbb{1} \\ \cap & & \cap \\ \mathcal{S}^+ & & H^+ . \end{array}$$

Cf. Proof of Thm 3.6

$$u_{A_n}(x) = G\,l_{A_n}(x)$$

$$= \int_{\hat{X}_M} K(x, \alpha)\, \mu_n(d\alpha)$$

$$\mu_n = \sum_{y \in X} G(o, y)\, l_{A_n}(y)\, \delta_y \longrightarrow \text{A repn measure of } \mathbb{1}.$$

Last exit time

Let $A \underset{\subset}{\ll} X$. We set

$$\tau_A(\omega) := \sup_{n \geq 0} \{ n \geq 0 \mid \omega_n \in A \}$$

$$= 0, 1, 2, \ldots +\infty . \qquad \text{measurable}.$$

with domain $\omega \in \Omega \setminus \prod_{n=0}^{\infty} A^c$

NOTE.

If $x \in A$, $\mathbb{P}_x\left(\prod_{n=0}^{\infty} A^c\right) = 0$.

Thus τ_A is well-defn. a.e. for $\mathbb{P}_x$, $x \in A$.

Rem. 4.6

- $\mathbb{P}_x(\tau_A = 0) = l_A(x) \qquad x \in A$.

(• $\mathbb{P}_x(Z_{\tau_A} = y) = K(x, y)\, G(o, y)\, l_A(y)$.)

$(\longrightarrow$ Thm 4.8 proof$)$

- $A_1 \subset A_2 \subset \cdots \nearrow X \Rightarrow \tau_{A_n} \nearrow +\infty$. $\mathbb{P}_x$-a.s.

§ 4.3 Harmonic Measures

(X, P) : irred & transient.

We will show Thm 3.11 (unique repn)

assuming Thm 4.3 (convergence to boundary)

$$\text{Thm } 4.3 \xrightarrow{\S 4.3.\cancel{4}} \text{Thm } 3.11$$

§ 4.~~4~~.

Thus now we have :

This phenomenon does not depend on ~~the~~ origin. →

$\forall x \in X$ for $\mathbb{P}_x$-a.e. $\omega \in \Omega$

$$\exists \lim_{n\to\infty} Z_n(\omega) \in \partial_M X.$$

We set $Z_\infty : \Omega \to \partial_M X$ measurable.

$$Z_\infty(\omega) := \lim_{n\to\infty} Z_n(\omega).$$

NOTE

$$\mathbb{P}_x(Z_\infty \text{ is not defined}) = 0 \quad \forall x \in X.$$

Defn 4.7

For each $x \in X$, put $\nu_x := Z_{\infty *}(\mathbb{P}_x) \in \mathrm{Prob}(\partial_M X)$

i.e.

$$\nu_x(A) = \mathbb{P}_x(Z_\infty^{-1}(A)), \quad A \subset \partial_M X$$

The family ν_x, $x \in X$ is called the harmonic measures

☆ $h(x) := \nu_x(A) \in H^+.$

$$Ph(x) = \mathbb{P}_x(\underset{\substack{\uparrow \\ \text{shift}}}{\sigma^{-1}}(Z_\infty^{-1}(A)))$$

Lem 4.4

ω_n converges $\Leftrightarrow$ ω_{n+1} converges

$$= \mathbb{P}_x(Z_\infty^{-1}(A))$$

$$= h(x).$$

☆ By defn, $\forall f : \partial_M X \to \mathbb{R}_+$ Borel.

$$\int_{\partial_M X} f(\alpha)\, \nu_x(d\alpha) = \int_\Omega f(Z_\infty(\omega))\, \mathbb{P}_x(d\omega)$$

$$\forall x \in X.$$

Thm. 4.8

$\forall x \in X \quad \forall o \in X$ origin

$\forall A \subset \partial_M X$ we have

$$\nu_x(A) = \int_A K(x,\alpha)\, \nu_o(d\alpha)$$

where $K(x,y) = \dfrac{G(x,y)}{G(o,y)} \quad x,y \in X.$

☆ $\dfrac{d\nu_x}{d\nu_o}(\alpha) = K(x,\alpha)$

☆ Put $A = \partial_M X$. Then we have

$$1 = \int_{\partial_M X} K(x,\alpha)\, \nu_o(d\alpha) \quad \forall x \in X$$

Integral repn of $\mathbb{1}$.

Proof.

STEP 1: We show ν_o represents $\mathbb{1}$.

Recall our potential approximation of $\mathbb{1}$:

$$A_1 \subset A_2 \subset \cdots \nearrow X.$$

$$u_{A_n}(x) = \mathbb{P}_x(Z_m \in A_n \text{ for some } m)$$
$$= (G\, \ell_{A_n})(x)$$

$u_{A_n}(x) \nearrow 1$. pt wise.

We know the w^*-limit of (a subsequence of)

$$\sum_{y \in X} G(o,y)\, \ell_{A_n}(y)\, \delta_y$$

converges to a measure representing $\mathbb{1}$. (Proof of Thm 3.6)

We will show its limit exactly equal ν_o.

Let τ_{A_n} = the last exit time as before

Then $\mathbb{P}_x(Z_{\tau_{A_n}} = y) = \sum_{m \geq 0} \mathbb{P}_x(\tau_{A_n} = m \ \& \ Z_m = y)$

$$= \sum_{m \geq 0} p_m(x, y)\, \ell_{A_n}(y)$$

$$= G(x, y)\, \ell_{A_n}(y)$$

Thus

$$\sum_{y \in X} G(x, y)\, \ell_{A_n}(y)\, \delta_y = \sum_{y \in X} \mathbb{P}_x(Z_{\tau_{A_n}} = y)\, \delta_y$$

$$\overset{?}{\downarrow}\ n \to \infty$$

$$\nu_x$$

Let $f \in C(\hat{X}_M)$ ~~& $x = 0$.~~

Then

$$\langle f, \sum_{y \in X} G(x, y)\, \ell_{A_n}(y)\, \delta_y \rangle$$

$$= \sum_{y \in X} f(y)\, \mathbb{P}_x(Z_{\tau_{A_n}} = y)$$

$$= \int_\Omega f(Z_{\tau_{A_n}})\, d\mathbb{P}_x \xrightarrow[\text{Lebesgue}]{n \to \infty} \int_\Omega f(Z_\infty)\, d\mathbb{P}_x = \int_{\partial_M X} f(\alpha)\, \nu_x(d\alpha)$$

Thus $\sum_{y \in X} G(x, y)\, \ell_{A_n}(y)\, \delta_y \xrightarrow[n \to \infty]{w^*} \nu_x$

Putting $x = 0$, we are done.

<u>STEP 2</u>. On ν_x.

$$\int_{\partial_M X} f\, d\nu_x = \lim_n \langle f, \sum_{y \in X} G(x, y)\, \ell_{A_n}(y)\, \delta_y \rangle$$

$$= \lim_n \sum_{y \in X} f(y)\, G(x, y)\, \ell_{A_n}(y)\, \delta_y$$

$$= \lim_n \sum_{y \in X} f(y)\, K(x, y)\, G(0, y)\, \ell_{A_n}(y)\, \delta_y$$

$$= \lim_n \langle f \cdot K(x, \cdot), \sum_y G(0, y)\, \ell_{A_n}(y)\, \delta_y \rangle$$

$$= \langle f \cdot K(x, \cdot), \nu_0 \rangle$$

$$= \int_{\partial_M X} f(\alpha)\, K(x, \alpha)\, \nu_0(d\alpha)$$

Let $h \in H^+(X, P)$.

$\rightsquigarrow$

(X, P^h) h-process

$\rightsquigarrow$

$(\Omega, \mathbb{P}_x^h)$

$$\exists Z_\infty = \lim_{n\to\infty} Z_n \in \partial_M X^h = \partial_M X$$
$(\mathbb{P}_x^h$ - a.s. $\omega)$

$\rightsquigarrow$

$$\nu_x^h := Z_{\infty *}(\mathbb{P}_x^h) \in \mathrm{Prob}(\partial_M X)$$

$\rightsquigarrow$

$$\nu_x^h(A) = \int_{\cancel{\partial_M X} A} K^h(x, \alpha)\, \nu_0^h(d\alpha)$$

$$= \int_{\cancel{\partial_M X} A} \frac{h(0) K(x,\alpha)}{h(x)}\, \nu_0^h(d\alpha)$$

Putting $A = \partial_M X$ we have the following

&

$$\mu_h := h(0)\, \nu_0^h$$

☆ $\mu_h(\partial_M X) = h(0)$. ☆ $1 = \int \frac{K(x,\alpha)}{h(x)} \mu_h(d\alpha) \Rightarrow \underset{P^h\text{-harmonic}}{K(\cdot,\alpha)} \in H_0^+$ μ_h-a.e.

Thm. 4.9

$$\boxed{\forall h \in H^+ \quad h(x) = \int_{\partial_M X} K(x,\alpha)\, \mu_h(d\alpha) \quad \forall x \in X}$$

NOTE

$\mu_h = h(0)\, \nu_0^h$ depends on 0.

STEP TOWARD Thm 3.11.

- $\mu_h(\partial_M X \setminus \partial_M^{min} X) = 0$
- uniqueness.

Lem. 4.10

$$\boxed{\text{If } h_1, h_2 \in H^+ \;\&\; h_1 \leq h_2, \text{ then } \mu_{h_1} \leq \mu_{h_2}}$$

Proof.

$$\mu_{h_1} = h_1(0)\, Z_{\infty *}(\mathbb{P}_0^{h_1}). \quad \mu_{h_2} = h_2(0)\, Z_{\infty *}(\mathbb{P}_0^{h_2})$$

$$h_2(0)\, \mathbb{P}_0^{h_2}(C_{a_0 \cdots a_n}) = \delta_{0,a_0}\, h_2(0)\, p^{h_2}(a_0, a_1) \cdots p^{h_2}(a_{n-1}, a_n)$$
$$= \delta_{0,a_0}\, \cancel{h_2(a_0)}\, p(a_0, a_1) \cdots p(a_{n-1}, a_n)\, \underset{\geq h_1(a_n)}{h_2(a_n)}$$
$$\geq h_1(0)\, \mathbb{P}_0^{h_1}(C_{a_0 \cdots a_n})$$

∎

Lem. 4.11

$\boxed{\begin{array}{l} \text{Suppose } \alpha \in \partial_M X \text{ satisfies } \underset{k_\alpha}{\underset{\|}{K(\cdot,\alpha)}} \in H_0^+ . \\ \text{Then } \alpha \in \partial_M^{min} X \iff \mu_{k_\alpha} = \delta_\alpha \end{array}}$

Proof.

$\Rightarrow$ Prop 3.9. is applied to

$$k_\alpha(\alpha) = \int_{\partial_M X} K(\alpha, \beta)\, \mu_{k_\alpha}(d\beta)$$

$\Leftarrow$ If $h \in H^+$ & $h \leq k_\alpha$.

Then Lemma $\rightarrow \mu_h \leq \mu_{k_\alpha} = \delta_\alpha$

$\rightarrow \mu_h = h(0)\, \delta_\alpha$

$$h(x) = \int_{\partial_M X} K(x, \beta)\, \mu_h(d\beta)$$

$$= h(0)\, \underset{k_\alpha(x)}{\underset{\|}{K(x, \alpha)}}.$$

Hence k_α is minimal ▨.

Lem.

☆ For $\alpha \in \partial_M^{min} X$. $(k_\alpha \in H_0^+)$.

$$\mu_{k_\alpha} = k_\alpha(0)\, \nu_0^{k_\alpha}$$

$$= Z_\infty(\mathbb{P}_0^{k_\alpha})$$

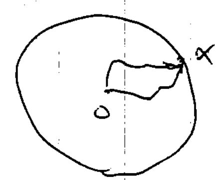
i.e.

$$\mu_{k_\alpha}(A) = \mathbb{P}_0^{k_\alpha}(Z_\infty \in A).$$

$$\| $$

$$\delta_\alpha(A).$$

Thus $\mathbb{P}_0^{k_\alpha}(Z_\infty = \alpha) = 1$



k_α — process.

☆ If $H^\infty(X, P) = \mathbb{C}\mathbb{1}$ i.e. $\underset{k_\alpha}{\underset{\|}{\mathbb{1}}} \in ex(H_0^+)$.

$$P^{k_\alpha} = P$$

$$\mathbb{P}_0(Z_\infty = \alpha) = 1$$

Lem. 4.12

$$\boxed{\forall h \in H^+ \quad \mu_h(\partial_M X \setminus \partial_M^{\min} X) = 0}$$

Proof.

$\varphi, \psi \in C(\hat{X}_M)$ $\quad h \in H_0^+$

$\left(\mu_h = h(0)\, \nu_0^h = Z_{\infty *}(\mathbb{P}_0^h) \right)$

$$\int_\Omega \varphi(Z_n)\, \psi(Z_{n+m})\, \mathbb{P}_0^h(d\omega) \quad (d\mathbb{P}_0^h)$$

$$= \sum_{x, y \in X} \varphi(x)\, \psi(y)\, p_n^h(0, x)\, p_m^h(x, y)$$

(n が消えた)

$$= \sum_{x \in X} \varphi(x)\, p_n^h(0, x) \int_\Omega \psi(Z_m)\, d\mathbb{P}_x^h$$

Letting $m \to +\infty$, we have

$$\int_\Omega \varphi(Z_n)\, \psi(Z_\infty)\, d\mathbb{P}_0^h$$

$$= \sum_{x \in X} \varphi(x)\, p_n^h(0, x) \int_\Omega \psi(Z_\infty)\, d\mathbb{P}_x^h$$

$$= \sum_{x \in X} \varphi(x)\, p_n^h(0, x) \int_{\partial_M X} \psi(\alpha)\, \nu_x^h(d\alpha).$$

$$= \sum_{x \in X} \varphi(x)\, p_n^h(0, x) \int_{\partial_M X} \psi(\alpha)\, K^h(x, \alpha)\, \nu_0^h(d\alpha)$$

$$= \int_{\partial_M X} \nu_0^h(d\alpha)\, \psi(\alpha) \sum_{x \in X} \varphi(x)\, \underbrace{p_n^h(0, x)\, K^h(x, \alpha)}_{= p_n(0,x)\, K(x,\alpha)}$$

$$= \int_{\partial_M X} \nu_0^h(d\alpha)\, \psi(\alpha) \sum_{x \in X} \varphi(x)\, p_n^{k_\alpha}(0, x)$$

(Recall $k_\alpha \in H_0^+$ ν_0^h-a.e. α)

$$= \int_{\partial_M X} \nu_0^h(d\alpha)\, \psi(\alpha) \int_\Omega \varphi(Z_n)\, d\mathbb{P}_0^{k_\alpha} \qquad (*)$$

Letting $n \to \infty$, we have

$$\int_\Omega \varphi(Z_\infty)\, \psi(Z_\infty)\, d\mathbb{P}_0^h = \int_{\partial_M X} \varphi(\alpha)\, \psi(\alpha)\, \underbrace{\nu_0^h(d\alpha)}_{= \mu_h}$$

$\| \; (*)$

$$\int_{\partial_M X} \nu_0^h(d\alpha)\, \psi(\alpha) \int_\Omega \varphi(Z_\infty)\, d\mathbb{P}_0^{k_\alpha} =$$

$$= \int_{\partial_M X} \nu_0^h(d\alpha)\, \psi(\alpha) \int_{\partial_M X} \varphi(\beta)\, \nu_0^{k_\alpha}(d\beta) \qquad \forall \varphi, \forall \psi$$

ν_0^h-a.e. α

$$\to \varphi(\alpha) = \int_{\partial_M X} \varphi(\beta)\, \nu_0^{k_\alpha}(d\beta) \quad \forall \varphi \;\to\; \mu_h\text{-a.e. } \alpha \quad \mu_{k_\alpha} = \delta_\alpha$$

$\Updownarrow$ Lem 4.11

$\alpha \in \partial_M^{\min} X$

No.

<u>Proof of Thm 3.11</u>

Let $h \in H_0^+$. We know.

$$h(x) = \int_{\partial_M X} K(x,\alpha)\ \mu_h(d\alpha)$$

$$= \int_{\partial_M^{\min} X} K(x,\alpha)\ \mu_h(d\alpha)$$

Suppose

$$h(x) = \int_{\partial_M^{\min} X} K(x,\alpha)\ \mu'(d\alpha) \quad -(*)$$

Want to show

$$\int_{\partial_M^{\min} X} f(\alpha)\ \mu_h(d\alpha) = \int_{\partial_M^{\min} X} f(\alpha)\mu'(d\alpha)$$

$$\forall f \in C(\hat{X}_M).$$

$$\int_{\partial_M^{\min} X} f(\alpha)\ \mu_h(d\alpha) = \int_\Omega f(Z_\infty)\ \mathbb{P}_0^h(d\omega)$$

$$= \lim_{n\to\infty} \int_\Omega f(Z_n)\ \mathbb{P}_0^h(d\omega)$$

$$= \lim_{n\to\infty} \sum_{y\in X} f(y)\ p_n^h(0,y)$$

$$= \lim_{n\to\infty} \sum_{y\in X} f(y)\ p_n(0,y)\ h(y)$$

$$\overset{(*)}{=} \lim_{n\to\infty} \sum_{y\in X} f(y)\ p_n(0,y) \int_{\partial_M^{\min} X} K(y,\alpha)\ \mu'(d\alpha)$$

$$= \lim_{n\to\infty} \int_{\partial_M^{\min} X} \Big(\sum_{y\in X} f(y)\ p_n(0,y)\ K(y,\alpha)\Big)\ \mu'(d\alpha)$$

$$= \lim_{n\to\infty} \int_{\partial_M^{\min} X} \Big(\sum_{y\in X} f(y)\ p_n^{k_\alpha}(0,y)\Big)\ \mu'(d\alpha)$$

$$= \lim_{n\to\infty} \int_{\partial_M^{\min} X} \Big(\int_\Omega f(Z_n)\ d\mathbb{P}_0^{k_\alpha}\Big)\ \mu'(d\alpha)$$

$$= \int_{\partial_M^{\min} X} \int_\Omega f(Z_\infty)\ d\mathbb{P}_0^{k_\alpha}\ \mu'(d\alpha)$$

$$= \int_{\partial_M^{\min} X} \Big(\int_{\cancel{\Omega}\,\partial_M X} f(\beta)\ \mu_{k_\alpha}(d\beta)\Big)\ \mu'(d\alpha)$$

$$= \int_{\partial_M^{\min} X} f(\alpha)\ \mu'(d\alpha).$$

∎

§4.4 Proof of Thm 4.3

★ Doob's supermartingale conv. thm.

$(\Omega, \mathcal{F}, \mathbb{P})$ prob. sp.

Let $\mathcal{F}_n \subset \mathcal{F}$ $(n \geq 0)$ be a filtration

i.e. $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$

σ-field.

Random variables $\{X_n : \Omega \to \mathbb{R}\ (n \geq 0)\}$

is supermartingale w.r.t. $(\Omega, \mathcal{F}, \mathbb{P})$

if

(1) X_n is $\mathcal{F}_n$-meas. $n \geq 0$

(2) $\int_\Omega |X_n|\, d\mathbb{P} < \infty$ $\quad n \geq 0$.

(3) $\int_\Omega X_{n+1} 1_A\, d\mathbb{P} \leq \int_\Omega X_n 1_A\, d\mathbb{P}$

$\forall A \in \mathcal{F}_n$.

★ (3) $\Leftrightarrow$ $\mathbb{E}(X_{n+1} \mid \mathcal{F}_{n+1}) \leq X_n$.

cond. exp.

★ (3), $=$ $\to$ martingale.

Ex. 4.13

- (X, P) : Markov chain (not necessarily irred)
- $f \in S^+(X, P)$

$\rightsquigarrow$ $f \circ Z_n : \Omega \longrightarrow \mathbb{R}$ is

path space

supermartingale w.r.t. $\mathcal{F}_n$ & $\mathbb{P}_\gamma$

any initial probability with $\sum_{x \in X} f(x)\gamma(x) < \infty$

(1) Trivial

(2) $$\int_\Omega f(Z_n(\omega))\, \mathbb{P}_x(d\omega) = \sum_{y \in X} f(y)\, p_n(x, y)$$

$$= (P^n f)(x)$$

$$\leq f(x) < \infty.$$

(3) It suffices to check for $A = C_{a_0 \cdots a_n} \in \mathcal{F}_n$.

$$\int_\Omega f(Z_{n+1}(\omega))\, 1_A\, \mathbb{P}_x(d\omega)$$

$$= \sum_{y\in X} \int_\Omega f(y)\, 1_A\, 1_{Z_{n+1}=y}\, \mathbb{P}_x(d\omega)$$

$$= \sum_{y\in X} f(y)\, \delta_{x,a_0}\, p(a_0 a_1)\cdots p(a_{n-1}, a_n)\, p(a_n, y)$$

$$= \delta_{x,a_0}\, p(a_0, a_1)\cdots p(a_{n-1}, a_n)\, (Pf)(a_n)$$

$$\leqq \quad \text{——}'' \text{——} \quad f(a_n)$$

$$= \int_\Omega f(Z_n(\omega))\, 1_A\, \mathbb{P}_x(d\omega)$$

Ω		(X, P)
supermartingale	$\longleftrightarrow$	superharm
martingale	$\longleftrightarrow$	harm.

Thm 4.1~~4~~ (Doob)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ prob sp.

Let X_n r.v. on Ω supermartingale w.r.t. a filter $\mathcal{F}_n$ $(n \geqq 0)$

Suppose $\sup_{n\geqq 0} \mathbb{E}(|X_n|) < \infty$.

Then for $\mathbb{P}$-a.s. $\omega \in \Omega$, the limit $\underset{(n\to\infty)}{\lim} X_n(\omega)$ exists $(= X_\infty)$

& $\mathbb{E}[|X_\infty|] < \infty$

The proof relies on Doob's upcrossing/down crossing ~~lemma~~ ~~inequality~~ lemma.

$\Uparrow$

For our proof of Thm 4.3,

this lemma is ~~essential~~ ~~more~~ essential.

(rather than the theorem above)

Rem.

If $X_n \geqq 0$ $\forall n \geqq 0$, then sup-condition is automatic because $\mathbb{E}(X_n) \underset{\text{supermart.}}{\leqq} \mathbb{E}(X_{n-1}) \leqq \cdots \leqq \underbrace{\mathbb{E}(X_0)}_{<\infty}$

★ Upcrossing / Down crossing numbers.

$a, b \in \mathbb{R}$ $a < b$ & $N \geq 0$.

Set the upcrossing number.

$$U_N^{[a,b]} : \mathbb{R}^{N+1} \to \mathbb{N}$$
$$(x_0, \dots x_N) \mapsto U_N^{[a,b]}(x_0, \dots x_N)$$

by

$U_N^{[a,b]}(x_0, \dots, x_N)$

:= the supremum of $k \geq 0$ such that

$\exists$ (times) $0 \leq t_1 < \cdots < t_{2k} \leq N$

satisfying

$x_{t_1} \geq b$, $x_{t_2} \leq a$

$x_{t_3} \geq b$, $x_{t_4} \leq a$

$\vdots$

$x_{t_{2k-1}} \geq b$. $x_{t_{2k}} \leq a$

Downcrosses だね!

いれかえ

The down crossing number is similarly defined

$D_N^{[a,b]}(x_0, \dots x_N)$

:= sup of $k \geq 0$ s.t.

$\exists\, 0 \leq t_1 < \cdots < t_{2k} \leq N$

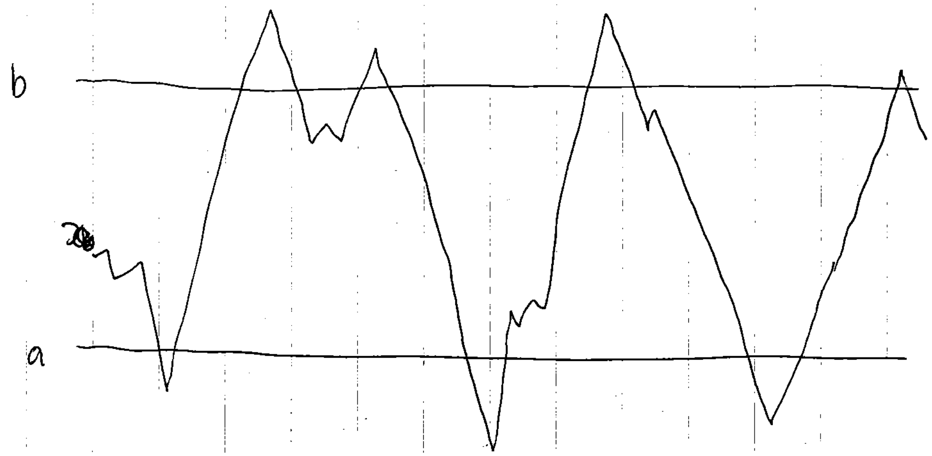
satisfying

$x_{t_1} \leq a$ $x_{t_2} \geq b$

$x_{t_3} \leq a$ $x_{t_4} \geq b$

$\vdots$

$x_{t_{2k-1}} \leq a$ $x_{t_{2k}} \geq b$.



Upcrossing nb = 3.

Down crossing = 2.

☆ Easy facts :

- $U_N^{[a,b]}(x_0, \dots x_N) = D_N^{[a,b]}(x_N, \dots x_0)$
- $\left| U_N^{[a,b]} - D_N^{[a,b]} \right| \leq 1$
- $U_N^{[a,b]}, D_N^{[a,b]} : \mathbb{R}^{N+1} \longrightarrow \mathbb{N}$ Borel.

<u>Lem</u>. 4.15 (Doob ~~down~~ up crossing lemma)

$$\boxed{\begin{array}{l} X_n \geq 0 . \quad (n \geq 0). \quad \text{supermartingale} \\ \mathbb{E}[D_N^{[a,b]}(X_0, \dots, X_N)] \leq \dfrac{\mathbb{E}[X_0]}{b-a} \end{array}}$$

D: downcrossing

<u>Proof of supermartingale conv.</u>

$$U^{[a,b]}(X_n) := \lim_{n \to \infty} U^{[a,b]}(X_0 \dots X_N)$$

$\to$ down crossing Lemma & monotone conv. thm $\forall a < \forall b$

$$\mathbb{E}[U^{[a,b]}(X_n)] \leq \frac{\mathbb{E}[X_0]}{b-a}$$

<u>大事なこと</u> $x_n \in \mathbb{R}_{\#}$ seq.

$$U^{[a,b]}(x_n) := \lim_N U_N^{[a,b]}(x_0 \dots x_N) \nearrow$$

$$\in \mathbb{N} \cup \{+\infty\}.$$

☆ $\lim_{n \to \infty} x_n \in \mathbb{R} \cup \{\pm\infty\} \iff \forall a, \forall b \in \mathbb{R}$ $U^{[a,b]}(x_n) < \infty$.

$\nexists \lim_{n \to \infty} x_n \iff \underset{\geq -\infty}{\underline{\lim}\, x_n} < \underset{\leq +\infty}{\overline{\lim}\, x_n}$

$\iff \exists a < \exists b \quad \underline{\lim}\, x_n < a < b < \overline{\lim}\, x_n$

$\iff \exists a, \exists b \quad U^{[a,b]}(x_n) = +\infty.$

Thus

$$U^{[a,b]}(X_n) < \infty \quad a.s.$$

$$\to \quad X_n \longrightarrow {}^{\exists}X_\infty \in \mathbb{R}_+ \cup \{+\infty\} \text{ a.s.}$$

Fatou

$$\rightsquigarrow \quad \mathbb{E}(X_\infty) \leqq \underline{\lim}\, \mathbb{E}(X_n) \overset{\text{sup. mart.}}{\leqq} \mathbb{E}(X_0) < +\infty.$$

$$\rightsquigarrow \quad X_\infty \in \mathbb{R}_+ \quad a.s. \qquad ▨$$

<u>Markov time</u>

$(\Omega, \mathcal{F}, \mathbb{P})$ prob. space.

$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ filtration.

$\tau : \Omega \to \mathbb{N}$ measurable

is a <u>Markov time</u> iff $\{\omega \mid \tau(\omega) = n\} \in \mathcal{F}_n$.

<u>NOTE</u>

- $\{\tau \leqq n\} \in \mathcal{F}_n \qquad \{\tau > n\} \in \mathcal{F}_n$

  $\{\tau \geqq n\} = \{\tau > n-1\} \in \mathcal{F}_{n-1}$

- σ, τ : Markov time $\Rightarrow$ $\sigma \vee \tau$, $\sigma \wedge \tau$ Markov.

$$\{\sigma \vee \tau = n\} = \bigcup_{k=0}^{n} \{\sigma = k\} \cap \{\tau = n\} \cup \bigcup_{k=0}^{n} \{\sigma = n\} \cup \{\tau = k\} \in \mathcal{F}_n$$

$\cancel{n \geqq k, \ell \geqq 0}$

$$\{\sigma \wedge \tau = n\} = \bigcup_{k=n}^{\infty} \{\sigma = k\} \cap \{\tau = n\} \cup \bigcup_{\ell=n}^{\infty} \{\sigma = n\} \cup \{\tau = k\}$$

$$= (\{\sigma \geqq n\} \cap \{\tau = n\}) \cup (\{\sigma = n\} \cap \{\tau \geqq n\}) \in \mathcal{F}_n.$$

- $\tau = \text{const} = k$ Markov.

$$\{\tau = n\} = \begin{cases} \phi & \text{if } n \neq k \\ \Omega & \text{if } n = k \end{cases}$$

For X_n : supermartingale & τ : Markov time.

$$X_\tau : \Omega \to \mathbb{R} \quad \text{r.v.}$$

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega), \quad \omega \in \Omega.$$

(measurable)

Lem. 4.16

$$\boxed{\begin{array}{l}\text{Let } \sigma, \tau \text{ Markov time } \& \ \sup_\omega \tau(\omega) < \infty. \\ \text{then} \quad \mathbb{E}[X_\tau] \leqq \mathbb{E}[X_\sigma]\end{array}}$$

Proof.

STEP 1. Let $n \leqq N$. For any , one has.
If τ : Markov time satisfies $n \leqq \tau \leqq N$.
Then

$$\mathbb{E}[1_A \overset{X_\tau}{\cancel{Z_\tau}}] \leqq \mathbb{E}[1_A \overset{X_n}{\cancel{Z_n}}] \quad {}^\forall A \in \mathcal{F}_n$$

By induction w.r.t. n from N.
When $n = N$. trivial
Suppose $n = N, N-1, \dots, m$ OK.
Let $A \in \mathcal{F}_{m-1}$

$$\mathbb{E}[1_A \overset{X_\tau}{\cancel{Z_\tau}}] = \mathbb{E}[1_A 1_{\{\tau=m-1\}} \overset{X_\tau}{\cancel{Z_\tau}} + \underbrace{1_A 1_{\{\tau>m-1\}} \overset{X_\tau}{\cancel{Z_\tau}}}_{= 1_A 1_{\{\tau>m-1\}} \overset{X_{\tau\vee m}}{\cancel{Z_{\tau\vee m}}}}]$$

$1_A 1_{\{\tau>m-1\}} \in \mathcal{F}_{m-1}$, $m \leqq \tau\vee m$.

$$\leqq \mathbb{E}[\underbrace{1_A 1_{\{\tau=m-1\}}}_{\in \mathcal{F}_{m-1}} \overset{X_{m-1}}{\cancel{Z_{m-1}}}] + \mathbb{E}[1_A 1_{\{\tau>m-1\}} \overset{X_m}{\cancel{Z_m}}]$$

$$\leqq \mathbb{E}[1_A 1_{\{\tau=m-1\}} \overset{X_{m-1}}{\cancel{Z_{m-1}}}] + \mathbb{E}[1_A 1_{\{\tau>m-1\}} \overset{X_{m-1}}{\cancel{Z_{m-1}}}] \quad \text{supermartingale.}$$

$$= \mathbb{E}[1_A X_{m-1}]$$

STEP 2. Let $\sigma \leqq \tau \leqq N$.

$$\mathbb{E}[\cancel{X_\tau}] = \sum_{n\geqq 0} \mathbb{E}[1_{\{\sigma=n\}} X_\tau]$$

$$= \sum_{n\geqq 0} \mathbb{E}[\underbrace{1_{\{\sigma=n\}}}_{\in \mathcal{F}_n} X_{\tau\vee n}]$$

$$\overset{\text{STEP 1}}{\leqq} \sum_{n\geqq 0} \mathbb{E}[1_{\{\sigma=n\}} X_n]$$

$$= \mathbb{E}[X_\sigma]$$

Proof of Lem 4.15

We will construct Markov times $\tau_0 \leqq \tau_1 \leqq \cdots$ inductively as follows

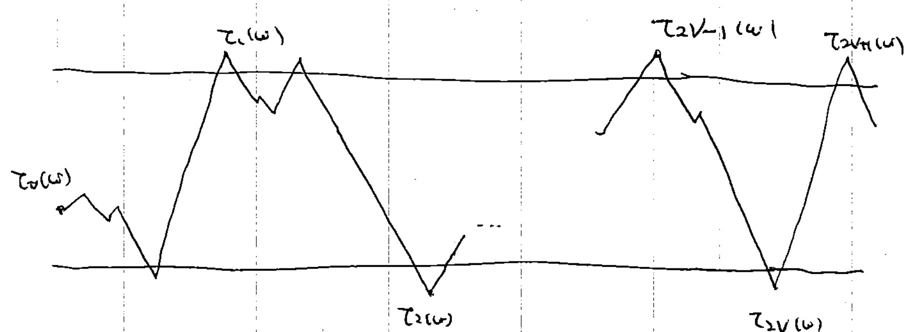
Put $\tau_0 = 0$.

For n: odd,

$$\tau_n(\omega) := \begin{cases} \inf\{k \in \mathbb{N} \mid \tau_{n-1}(\omega) \leqq k \leqq N,\ X_k(\omega) \geqq b\} \\ N \quad \text{if } \{\ \} = \phi \end{cases}$$

n: even

$$\tau_n(\omega) := \begin{cases} \inf\{k \in \mathbb{N} \mid \tau_{n-1}(\omega) \leqq k \leqq N,\ X_k(\omega) \leqq a\} \\ N \quad \text{if } \{\ \} = \phi \end{cases}$$



$$\nu(\omega) := U^{[a,b]}(X_0(\omega), \ldots X_N(\omega))$$

if $n \geqq 2\nu + 2$ $\tau_n(\omega) = N$.

Now let

$$S := \sum_{k=0}^{\nu} (X_{\tau_{2k+1}} - X_{\tau_{2k+2}})$$

Then since $X_{\tau_{2k+1}} \geqq b$ & $X_{\tau_{2k}} \leqq a$

$$S \geqq \nu \cdot (b-a).$$

On the other hand.

$$S = X_{\tau_1} + \sum_{k=0}^{\nu-1} (X_{\tau_{2k+3}} - X_{\tau_{2k+2}}) - X_{\tau_{2\nu+2}}$$

Then

$$\mathbb{E}[S] = \mathbb{E}[X_{\tau_1}] + \sum_{k=0}^{\nu-1} \Big(\underbrace{\mathbb{E}[X_{\tau_{2k+3}}] - \mathbb{E}[X_{\tau_{2k+2}}]}_{\leqq 0 \text{ Lem.}}\Big) - \mathbb{E}[X_{\tau_{2\nu+2}}]$$

$$\leqq \mathbb{E}[X_{\tau_1}] \leqq \mathbb{E}[X_0].$$

Hence

$$(b-a)\,\mathbb{E}[\nu] \leqq \mathbb{E}[S] \leqq \mathbb{E}[X_0]$$

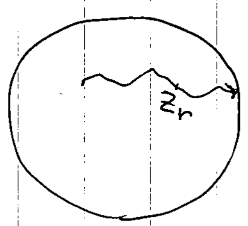
∎

Recall : want to prove

(X, P) irr & trans.

$\forall \gamma \in \mathrm{Prob}(X)$ initial distr.

$$X \ni Z_n(\omega) \xrightarrow[n\to\infty]{} {}^{\exists} Z_\infty(\omega) \in \partial_M X \quad \mathbb{P}_\gamma\text{-a.s.}\,\omega.$$



☆ i.e. we must show

$$f(Z_n(\omega)) \xrightarrow[n\to\infty]{} f(Z_\infty(\omega))$$

$$\forall f \in C(\hat{X}_M) = C_0(X) + \overset{*}{C}(\ ).$$

☆ If $f \in C_0(X)$ then $f(Z_n(\omega)) \to 0$.

WMA $f = \delta_x$ $(x \in X)$

$\{\omega \mid f(Z_n(\omega)) \not\to 0\} = \{\omega \mid Z_n(\omega) = x \text{ inf many } n\}$

probability 0. by trans. (Thm 4.5).

So, our aim is to prove.

$$K(x, Z_n(\omega)) \xrightarrow[n\to\infty]{} \text{converges}. \quad \forall x \in X$$

$\mathbb{P}_\gamma$ - a.s. ω.

NOTE.

$K(x, \cdot)$ is not P-superharmonic

→ We cannot apply Doob's conv. thm directly!

IDEA Take the dual of P.

Let

$\eta(x) := G(o, x)$ ← row vector

$\hat{P} := (\hat{p}(x, y))_{x, y \in X}$

$\hat{p}(x, y) = \eta(y)\, p(y, x)\, \eta(x)^{-1}$

Then

- $\displaystyle\sum_{y\in X} \hat{p}(x,y) = \sum_{y\in X} \eta(y)\, p(y,x)\, \eta(x)^{-1}$

$$= (GP)(0,x)\ \eta(x)^{-1}$$

$$\leq G(0,x)\ \eta(x)^{-1} = 1.$$

inequality

- $\displaystyle\sum_{y\in X} \hat{p}(x,y)\, K(z,y) = \sum_{y\in X} \eta(y)\, p(y,x)\, \underbrace{\eta(x)^{-1}}_{}$ $K(z,y)$

$$= \sum_y \cancel{\eta}\, G(0,y)\, p(y,x)\, G(0,x)^{-1} \quad \frac{G(z,y)}{G(0,y)}$$

$$= (GP)(z,x)\ G(0,x)^{-1}$$

$$\leq \frac{G(z,x)}{G(0,x)} = K(z,x).$$

Hence. $K(z,\cdot)$ is $\hat{P}$ - superharmonic.

Take the grave yard extension:

$$\tilde{X} := X \cup \{*\}$$

$$Q := \begin{array}{c|cc} & X & * \\ \hline X & \hat{P} & \mathbb{1}-\hat{P}\mathbb{1} \\ * & 0 & 1 \end{array}$$

$\star$ If $f : X \to \mathbb{R}_+$ $\hat{P}$-sup.harmonic

$\hat{f} : \tilde{X} \to \mathbb{R}_+$ Q-sup.harmonic

putting $\hat{f}(*) = 0$.

$$Q\hat{f} = \begin{bmatrix} \hat{P}f \\ 0 \end{bmatrix} \leq \hat{f}.$$

Hence. on $\Omega_Q := \tilde{X}^{\mathbb{N}}$ path space.

$\forall \nu \in \mathrm{Prob}(\tilde{X})$. initial distr.

$\hat{f}(\omega_n) \longrightarrow \exists \lim_{n\to\infty} \hat{f}(\omega_n)$

$\mathbb{P}_\nu$ - a.s.w.

NOTE.

This does not imply $f(\omega_n) \to \exists \lim_{n\to\infty} f(\omega_n)$ a.s. $\omega \in \Omega_p$.

Because.

$$\mathbb{P}_\nu^{\Omega_Q}(C_{a_0\cdots a_n}) = \nu(a_0)\, Q(a_0 a_1) \cdots Q(a_{n-1} a_n)$$
$(a_k \in X)$

$$= \nu(a_0)\, \hat{p}(a_0 a_1) \cdots \hat{p}(a_{n-1} a_n)$$

$$= \nu(a_0)\, \eta(a_0)^{-1} p(a_1 a_0) \cancel{\eta(a_1)}$$
$$\cancel{\eta(a_1)^{-1}}\, p(a_2 a_1) \cancel{\eta(a_2)}$$
$$\vdots$$
$$\cancel{\eta(a_{n-1})^{-1}}\, p(a_n a_1)\, \eta(a_n)$$

$$= \mathbb{P}_\eta^{\Omega_p}(C_{a_n a_1 \cdots a_0})\, \underbrace{\nu(a_0)\eta(a_0)^{-1}}_{?} \quad \text{—} \cancel{(*)}$$

unbdd measure

reverse.

Now, we ~~consider~~ make use of the ~~final~~ last exit time.

$A \subset\subset X$.

$\tau_A : \Omega_p \to \mathbb{N} \cup \{+\infty\}$.

$$\tau_A(\omega) = \sup_{n\geq 0} \{ n \mid \omega_n \in A \}$$

$$\text{Domain} = \bigcup_n \{ \omega \mid \omega_n \in A \}.$$

Recall its properties:

- $\mathbb{P}_x(\tau_A = 0) = l_A(x)$
- $\mathbb{P}_x(Z_n \in A \text{ for some } n \geq 0) = (G l_A)(x)$. (Comments after Thm 4.5)
- $\mathbb{P}_x(Z_{\tau_A} = y) = K(x, y)\, G(o, y)\, l_A(y)$.

Proof

Now we put $\nu \in \cancel{\mathrm{Prob}}\ \mathrm{Meas}(\tilde{X})$

$$\nu(s) = \eta(s)\, l_A(s) \qquad s \in X.$$

l_A depending on A

$$\nu(*) = 0.$$

$$\star \quad \sum_{s \in \tilde{X}} \nu(s) = \sum_{s} \underbrace{\eta(s)}_{= G(0,s)} l_A(s) = G l_A(0) \leq 1.$$

Now we have the Down crossing nb ineq:

$$\mathbb{E}_\nu^{\Omega Q}\left[D_N^{[c,d]}(\hat{f}\circ Z_0, \ldots \hat{f}\circ Z_N)\right] \leq \frac{\mathbb{E}_\nu^{\Omega Q}[\hat{f}\circ Z_0]}{d-c} \qquad \text{— } (*)$$

$$\mathrm{RHS}\ \cancel{f}\ (*) = \frac{1}{d-c} \sum_{x \in X} f(x)\, \nu(x)$$

$$= \frac{1}{d-c} \sum_{x \in X} f(x)\, \eta(x)\, l_A(x).$$

if $f(x) = K(z, x)$

then

$$\frac{1}{d-c} \sum_{x \in X} \underbrace{K(z,x)\, G(0,x)}_{= G(z,x)} l_A(x) = \frac{G l_A(z)}{d-c} \leq \frac{1}{d-c}.$$

$\frac{1}{d-c}$ does not dep on A ~~& N~~.

LHS of $(*)$

In what follows,

$$D_N := D_N^{[c,d]}$$

$$U_N := U_N^{[c,d]}$$

$a_0 \cdots a_N \in \mathbb{Z}$ として X の元のみ とってよい

$$\geq \sum_{\substack{a_0 \cdots a_N \in X \\ D_N(f(a_0) \cdots f(a_N)) = k}} k\, \mathbb{P}_\nu^{\Omega Q}(C_{a_0 \cdots a_N})$$

$$= \sum_{\substack{a_0 \cdots a_N \in X \\ D_N(f(a_0) \cdots f(a_N)) = k}} k\, \mathbb{P}_\eta^{\Omega P}(C_{a_N \cdots a_0})\, \nu(a_0)\, \eta(a_0)^{-1}$$

文字を変更

$$= \sum_{\substack{a_0 \cdots a_N \in X \\ D_N(f(a_N) \cdots f(a_0)) = k}} k\, \mathbb{P}_\eta^{\Omega P}(C_{a_0 \cdots a_N})\, \underbrace{\nu(a_N)\, \eta(a_N)^{-1}}_{= l_A(a_N)}$$

$$= \sum_{\substack{a_0 \cdots a_N \in X \\ U_N(f(a_0) \cdots f(a_N)) = k}} k\, \underbrace{\mathbb{P}_\eta^{\Omega P}(C_{a_0 \cdots a_N})\, l_A(a_N)}. \qquad \text{— } (**)$$

We interpret this term.

$$\mathbb{P}_x^{\Omega_p}(Z_{\tau_A - N} = a_0, \ \dots, \ Z_{\tau_A} = a_N)$$

$$= \sum_{m \geqq N} \mathbb{P}_x^{\Omega_p}(\tau_A = m \ \& \ Z_{m-N} = a_0 \cdots Z_m = a_N)$$

$$= \sum_{m \geqq N} p_{m-N}(x, a_0)\, p(a_0, a_1) \cdots p(a_{N-1}, a_N)\, \underset{A}{\ell}(a_N)$$

$$= G(x, a_0)\, p(a_0, a_1) \cdots p(a_{N-1}, a_N)\, \ell_A(a_N) \quad \text{---}(***)$$

Thus

$$\text{LHS of } (*) \overset{(**) \ \& \ (***) \ \cancel{(\text{putting } x = 0.)}}{\geqq}$$

$$\geqq \sum_{\substack{a_0 \cdots a_N \in X \\ \mho_N(f(a_0) \cdots f(a_N)) = k}} k \ \mathbb{P}_0^{\Omega_p}(Z_{\tau_A - N} = a_0, \cdots Z_{\tau_A} = a_N)$$

$$= \mathbb{E}_0^{\Omega_p}\left(\mho_N^{[c,d]}\left(\tilde{f}(Z_{\tau_A - N}) \cdots \tilde{f}(Z_{\tau_A})\right)\right)$$

Hence we have

$$\mathbb{E}_0^{\Omega_p}\left(\mho_N^{[c,d]}\left(\tilde{f}(Z_{\tau_A - N}) \cdots \tilde{f}(Z_{\tau_A})\right)\right) \leqq \frac{1}{d-c}$$

We keep.

$(\ f = K(x, \cdot)\ ) \qquad \forall N = 0, 1, 2, \cdots$

If $\omega \in \Omega_p$, $\tau_A(\omega) - k < 0$ then $Z_{\tau_A(\omega) - k}(\omega) = *$.

Letting $N \to +\infty$. we have

$$\mathbb{E}_0^{\Omega_p}\left[\mho_{\tau_A}^{[c,d]}\left(f(Z_0), \cdots f(Z_{\tau_A})\right)\right] \leqq \frac{1}{d-c}$$

$\forall A \subset\subset X$.

NEXT putting $A = A_n$, $A_1 \subset A_2 \subset \nearrow X$.

Then $\tau_{A_n} \nearrow +\infty$ a.s. & monotone conv. thm

$$\mathbb{E}_0^{\Omega_p}\left[\mho^{[c,d]}(f(Z_n))\right] \leqq \frac{1}{d-c}$$

$\forall c < \forall d$.

Hence $f(Z_n) \longrightarrow \exists \lim_{n \to \infty} f(Z_n) \quad \mathbb{P}_0 - \text{a.s.}\ \omega$.

Hence. we have shown :

$$Z_n(\omega) \longrightarrow Z_\infty(\omega) \qquad \mathbb{P}_0-\text{a.s. } \omega.$$
$$\begin{matrix} \Cap & & \Cap \\ X & & \hat{X}_M \end{matrix}$$

i.e. $\exists \lim_{n\to\infty} K_o(\cancel{Z}\, x, Z_n(\omega))$.

$\Longleftrightarrow \exists \lim_{n\to\infty} K_{o'}(x, Z_n(\omega))$. $\mathbb{P}_0$-a.s. ω.

(Because the top. sp $\hat{X}_M$ does not dep on $\underline{o}$).

~~We know $\exists \lim_{n\to\infty} K_{o'}(x, Z_n(\omega))$ $\mathbb{P}_{o'}$ -a.s. ω.~~

Hence we are done. ∎

$v_A(x) := \mathbb{P}_x(\exists \lim_{n\to\infty} Z_n \in \partial_M X)$

this set is σ-inv

$\to v_A \in \mathcal{H}^+$. & $v_A(o) = 1$

$\xrightarrow[\text{max. principle}]{} v_A \equiv 1$.

∎