2018 @東大

1. Preliminary

- Notation general.
- End(M) etc
- C^*-tensor category

  $X \otimes Y$. $\bar{X}$ $d(X)$ etc.

  amenability.

2. Actions of $\mathcal{C}$.

- . $\mathcal{C}$ denotes a C^*-...
- . M denotes a prop inf vN.
- . $\mathcal{C} \overset{(\alpha, c)}{\curvearrowright} M$ cocycle action.

- Centrally freeness.

  . local quantize.

3. Rohlin property.

- construction.

4. Classification.

- 2-coh. vanishg type result in M^ω
- 1-coh. — " —
- Interesting argument.

5. (Amenable) subfactors.

- ~~Classification~~
- C^*-2-category & std inv.
- Classification.

$\pi_Z \cdot \alpha_Z = \beta_Z \cdot \pi_0$.

$X < Z$. $T : X \to Z$.

$$T^\beta T^{*\beta} \underset{\|}{\pi_Z(\alpha_Z(x))} = T^\beta \overset{\pi_X(\cancel{\alpha_X}(x))}{\overset{\|}{\beta_X(\pi_0(x))}} T^{\beta^*}$$

$$\pi_Z(T^\alpha T^{\alpha^*} \alpha_Z(x)) = \pi_Z(T^\alpha \alpha_X(x) T^{\alpha^*}) = \cancel{\pi_Z(T^\alpha)\, \beta_X(\pi_0(x))\, \pi_Z(T^{\alpha^*})}$$

$$\underset{\cancel{\mathbb{C}}}{\underset{\cap}{\underline{T^{\beta^*}\pi_Z(T^\alpha)}}} \in \beta_X(M_0)' \cap M = \mathbb{C}$$

E_X^α unique

$\alpha_X(M) \subset M \otimes B(H_X)$

$(t \cdot tr_X) \cdot E_X^\alpha = \tau \cdot tr_X$

$$\underset{\tau \cdot \phi_{\alpha_X}}{\underset{\|}{\alpha_X(\tau)}} = (\tau \cdot tr_X)(\cancel{h_X} \cdot 1)$$

$$\tau(\theta_{\bar X}(x)) = \underset{\tau(\underset{1}{\underset{\|}{\theta_{\bar X}(h)}}\, \theta_{\bar X}(x))}{\underset{\|}{\tau(hx)}}$$

Section 1. Preliminary

§1.1 General notations

- M : vN alg with sep M_*

$Z(M) := M' \cap M$.

- M_* is an M-M-bimodule :

$a \varphi b\,(x) := \varphi(bxa) \qquad x \in M.$

$a, b \in M. \quad \varphi \in M_*$

- $\varphi \in M_*^+$

$M_\varphi := \{ a \in M \mid a\varphi = \varphi a \}$

the centralizer of φ

- vN subalg of M
- $\varphi \restriction_{M_\varphi}$ tracial.
- $Z(M) \subset M_\varphi$

- For $x \in M$,

$|x|_\varphi := \varphi(|x|)$

$\|x\|_\varphi := \varphi(x^* x)^{\frac{1}{2}}$ ← triangle ineq. ok.

- $a \in M_\varphi \qquad \|a\varphi\| \leqq |a|_\varphi$

$\|\varphi a\| \leqq |a|_\varphi$

(∵) Let $a = v|a|$. p.d.

$^\forall x \in M$

$a\varphi(x) = \varphi(xa) = \varphi(xv|a|)$

$= \varphi(|a|^{\frac{1}{2}} x v |a|^{\frac{1}{2}}) \qquad |a|^{1/2} \in M_\varphi.$

$\leqq \varphi(|a|)^{\frac{1}{2}} \varphi(|a|^{\frac{1}{2}} \underset{\leqq \|x\|^2 \cdot 1}{\boxed{v^* x^* x v}} |a|^{\frac{1}{2}})^{\frac{1}{2}}$

$\leqq \|x\| \varphi(|a|)$. ∎

- $|a+b|_\varphi \leqq |a|_\varphi + |b|_\varphi \qquad {}^\forall a, b \in M_\varphi$

(∵) $|a+b|_\varphi = \varphi(u^*(a+b)) \qquad u|a+b| = a+b$

$= \varphi(u^* a) + \varphi(u^* b)$

$\leqq |a|_\varphi + |b|_\varphi$ ∎

- $\mathrm{Mor}(N, M) := \{\rho \mid \rho : N \to M$ unital faithful normal $*$-homo $\}$

(possibly non-sep)

For $\rho, \sigma \in \mathrm{Mor}(N, M)$

$(\rho, \sigma) := \{ a \in M \mid a\rho(x) = \sigma(x) a \ \forall x \in N \}$

The intertwiner sp from ρ to σ.

- $\rho : N \to M$ & $\sigma : M \to N$ are a conjugate pair

if $\exists\, a \in (\underset{\widehat{N}}{\mathrm{id}_N}, \sigma\rho)$ $\exists\, b \in (\underset{\widehat{M}}{\mathrm{id}_M}, \rho\sigma)$

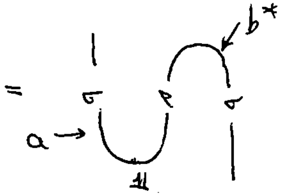
s.t.

$\sigma(b^*) a = \underset{\text{scalar}}{\underset{\neq 0}{c}} 1_N$ & $\rho(a^*) b = c 1_M$.

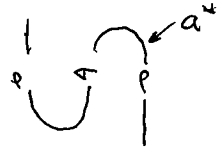
$*$ $\sigma(b^*) a \in (\sigma, \sigma)$, $\rho(a^*) b \in (\rho, \rho)$

$\sigma(b^*) a \sigma(x) = \sigma(b^*) \sigma\rho\sigma(x) a = \sigma(b^* \rho\sigma(x)) a$

$= \sigma(x) \sigma(b^*) a$

$\sigma(b^*) a =$


$= c\,1$

$\rho(a^*) b =$


$= c\,1$.

$\mathrm{Mor}(N, M)_0 := \{ \rho \in \mathrm{Mor}(N, M) \mid \rho$ has a conj map $\}$.

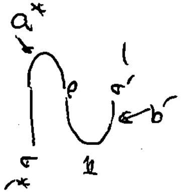
$*$ A conj of ρ is unique up to Ad unitary.

(∵) $a \in (\mathrm{id}_N, \sigma\rho)$ $b \in (\mathrm{id}_M, \rho\sigma)$

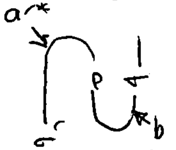
$= c1$.

$a' \in (\mathrm{id}_N, \sigma'\rho)$, $b' \in (\mathrm{id}_M, \rho\sigma')$

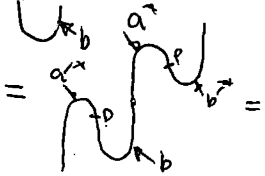
$= c'1$.

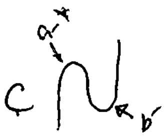
$a^* \sigma(b') =$


$\in (\sigma, \sigma')$

$a^* \sigma'(b) =$


$\in (\sigma', \sigma)$.

$a^* \sigma(b') a'^* \sigma'(b) =$


$= c$


$= cc' \neq 0$.

$\boxtimes$

$\mathrm{End}(M) := \mathrm{Mor}(M, M)$

$\mathrm{End}(M)_0 := \mathrm{Mor}(M, M)_0$

§1.2. Ultraproducts

$\mathbb{N} = \{1, 2, \cdots\}$ ← discrete top. sp.

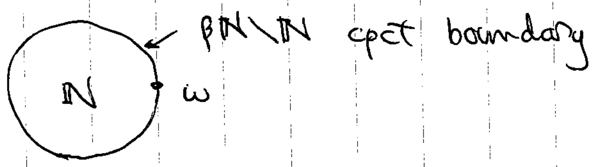
$\ell^\infty(\mathbb{N}) = C(\underset{\uparrow}{\beta\mathbb{N}})$

Gelfand spectrum

$\{\chi \mid \underset{\substack{*\text{-homo} \\ \chi(1) = 1}}{\chi : \ell^\infty(\mathbb{N}) \to \mathbb{C}}\}$

•
$$\begin{tikzcd} \mathbb{N} \arrow[r, hook] & \beta\mathbb{N} \\ n \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & ev_n \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$
$ev_n(x) := x_n$.

dense range



Fix $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$

Let $\underset{\mathbb{N}}{\overset{}{n}} \to \omega$. $\forall g \in \ell^\infty(\mathbb{N}) = C(\beta\mathbb{N})$

$g(n) \to g(\omega)$. by continuity.

⇝ $z_n \in$ ~~A~~ $\overset{\text{cpct Haus.}}{A}$ $n = 1, 2, \cdots$ $\exists! \lim_{n\to\omega} z_n \in A$.

(∵) $\forall f \in C(A)$. $\pi(f) := (f(z_1), f(z_2), \cdots) \in \ell^\infty(\mathbb{N})$

$f(z_n) \to \pi(f)(\omega)$.
$(n \to \omega)$

$f \mapsto \pi(f)(\omega)$ character on $C(A)$

• Ultraproduct vN alg.

M : vN alg

$\ell^\infty := \{(x_n) \mid x_n \in M.\ \sup\|x_n\| < \infty\}$ C^*-alg $\ni 1$

$J^\omega := \{(x_n) \in \ell^\infty \mid x_n \xrightarrow{s^*} 0\ (n \to \omega)$
i.e.
$x_n\varphi \xrightarrow{\|\cdot\|} 0\ \&\ \varphi x_n \xrightarrow{\|\cdot\|} 0\ \}$
$\forall\varphi \in M_*$

$\mathcal{M}^\omega := \{\underset{x}{(x_n)} \in \ell^\infty \mid xJ^\omega + J^\omega x \subset J^\omega\}$

multiplier alg of J^ω

$\mathcal{C}^\omega := \{(x_n) \in \ell^\infty \mid x_n\varphi - \varphi x_n \underset{n\to\omega}{\longrightarrow} 0$
$\forall\varphi \in M_*\}$

⇝ $J^\omega \subset \mathcal{C}^\omega \subset \mathcal{M}^\omega \subset \ell^\infty$ (norm closed)

$M^\omega := \mathcal{M}^\omega / \mathcal{J}^\omega$ $\quad (x_n)^\omega := (x_n) + \mathcal{J}^\omega$

$M_\omega := C^\omega / \mathcal{J}^\omega$ $\quad$ ultraproducts.

- $$\begin{tikzcd} M \arrow[r, hook] & M^\omega \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & (x, x, \dots)^\omega \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$ const embedding

$\rightsquigarrow$ $M_\omega \subset M' \cap M^\omega$

$(\because)$ $(x_n)^\omega \in M_\omega$ $\quad y \in M$.

$(x_n y - y x_n)^\omega \overset{?}{=} 0$

$x_n y \cdot \varphi \approx (y\varphi) \cdot x_n = y \varphi x_n \approx y x_n \varphi$ $\boxed{\equiv}$

- $$\begin{tikzcd} \tau^\omega : M^\omega \arrow[r] & M \\ (x_n)^\omega \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & \lim\limits_{n\to\omega} x_n \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

σ-weak convergence $\uparrow$ Alaoglu Thm. $M = (M_*)^*$.

faithful cond exp.

- For $\varphi \in M_*$ $\quad \varphi^\omega := \varphi \cdot \tau^\omega : M^\omega \to \mathbb{C}$

$$\varphi^\omega((x_n)^\omega) = \varphi(\lim_{n\to\omega} x_n)$$
$$= \lim_{n\to\omega} \varphi(x_n).$$

- In fact. M^ω $\quad W^*$-alg. s.t.

$\tau^\omega : M^\omega \to M$ faithful <u>normal</u> cond.exp.

M_ω $\quad W^*$-alg

since for $\forall \varphi \in M_*^+$ faithful state

$M_\omega = (M' \cap M^\omega)_{\varphi^\omega}$ $\boxtimes$

$(\because)$ $\subset$ ok

$\supset$ $(x_n)^\omega \in$ RHS.

$M_* \supset M\varphi$ $\quad$ norm dense

$x_n\, y\varphi \approx y x_n \varphi$

- $\forall \varphi . \psi \in M_*$ $\quad \forall x . y \in M^\omega$

$$\| x\varphi^\omega - \psi^\omega y \| = \lim_{n\to\omega} \| x_n \varphi - \psi y_n \|$$

$\rightsquigarrow$ $x \in M_\omega \Leftrightarrow x\varphi^\omega = \varphi^\omega x$ $\quad \forall \varphi \in M_*$

§ 1.3. C^*-tensor category

$\mathcal{C}$: C^*-tensor category

C^*-category with $\otimes$

$\overset{\text{defn}}{\Leftrightarrow}$

(1) $\overset{\text{objects}}{\forall X, Y \in \mathcal{C}}$ $\quad \overset{\text{sp of morphisms } T: X \to Y.}{\mathcal{C}(X,Y)}$ is a Banach sp.

$$\begin{array}{ccc} \mathcal{C}(Y,Z) \times \cancel{\mathrm{Mor}}\,\mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(X,Z) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (S,T) & \longmapsto & ST \end{array}$$

is bilinear & $\|ST\| \leqq \|S\|\|T\|$,

$$X \xrightarrow{T} Y \xrightarrow{S} Z$$

(2) $*$-operations on $\mathcal{C}(X,Y)$.

$$\begin{array}{ccc} * : \mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}(Y,X) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ T & \longmapsto & T^* \end{array}$$

- conj. linear
- $(ST)^* = T^* S^*$. & $T^*T \in \mathcal{C}(X,X)$ $T: X \to Y$ C^*-alg positive operator
- $T^{**} = T$
- $\|T^*T\| = \|T\|^2$.

(3) (Tensor products)

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$: bilinear bifunctor.

$\mathbb{1} \in \mathcal{C}$: tensor unit.

s.t.

- $\forall X, Y \in \mathcal{C}$ we have $X \otimes Y \in \mathcal{C}$.
- $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ (strictness) ~~trif~~ identity (associator)
- $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$.
- $S : X \to Y$. $T : Z \to W$

$\rightsquigarrow S \otimes T : X \otimes Z \to Y \otimes W$.

(We can ~~compute~~ treat $S \otimes T$ like tensor product C^*-algs e.g. $(S \otimes T)(S' \otimes T') = SS' \otimes TT'$)

- $(S \otimes T)^* = S^* \otimes T^*$.

(4) (Idempotent complete)

$\forall p \in \mathrm{End}(X)$ $\exists Y \in \mathcal{C}$ $\exists v : Y \to X$

s.t.

$v^*v = 1_Y$. $vv^* = p$ we denote $Y \prec X$

(5) (Direct sum).

$\forall X.\ \forall Y \in \mathcal{C}$. $\exists Z \in \mathcal{C}$ $\exists v : X \to Z$ s.t.

$w : Y \to Z$

$v^*v = 1_X$ $w^*w = 1_Y$. $vv^* + ww^* = 1_Z$.

(6) ($\mathbb{1}$ is simple)

$\mathrm{End}(\mathbb{1}) = \mathbb{C}1$

☆ $X \in \mathcal{C}$ <u>simple</u> if $\mathrm{End}(X) = \mathbb{C}1_X$ ~~Th~~

Defn 11 (Tensor functors)

$\mathcal{C}.\ \mathcal{D}$: C^*-tensor ~~fu~~ categories

$(F, L) : \mathcal{C} \to \mathcal{D}$ <u>unitary tensor functor</u>

if $F : \mathcal{C} \to \mathcal{D}$ functor between C^*-cats

- $X \mapsto F(X)$, $F(\mathbb{1}) = \mathbb{1}$.
- $T : X \to Y$ $\mapsto F(T) : F(X) \to F(Y)$. linear map

unitary → $\boxed{F(S^*) = F(S)^*}$ $F(ST) = F(S)F(T)$

$L_{X,Y} : F(X \cdot Y) \to F(X) \otimes F(Y)$ unitary.

<u>natural</u> w.r.t. $X . Y \in \mathcal{C}$

↑

(natural means the well-behavior when chaging variables X, Y)

i.e. if $T : X \to Z$ $S : Y \to W$

$$\begin{tikzcd} F(X \cdot Y) \arrow[r, "F(T \otimes S)"] \arrow[d, "L_{X,Y}"'] & F(Z \cdot W) \arrow[d, "L_{Z.W}"] \\ F(X) \cdot F(Y) \arrow[r, "F(T) \cdot F(S)"] \arrow[ur, phantom, "\circlearrowleft"] & F(Z) \cdot F(W) \end{tikzcd}$$

$L_{X,Y}$ satisfies the 2-cocycle rel.

$$(L_{X.Y} \otimes 1_{F(Z)}) L_{X \cdot Y, Z} = (1_{F(X)} \cdot L_{Y.Z}) \cdot L_{X, Y \cdot Z}$$

~~$F(X \cdot Y) \otimes F(Z)$~~

$$\begin{tikzcd} F(X \cdot Y \cdot Z) \arrow[r] \arrow[d, "L_{X \cdot Y. Z}"] & F(X) \otimes F(Y \cdot Z) \arrow[dd, "1_{F(X)} \cdot L_{Y.Z}"] \\ F(X \cdot Y) \otimes F(Z) \arrow[d, "L_{X,Y} \cdot 1_{F(Z)}"] & \\ F(X) \cdot F(Y) \cdot F(Z) & \end{tikzcd}$$

Ex. 1.2

Hilb_f : the collection of f.d. Hilb sps

$H, K \in \underset{\mathcal{C}}{\overset{\shortparallel}{\mathrm{Hilb}_f}}$ $\quad \mathcal{C}(H,K) = B(H,K)$.

$\mathbb{1} = \mathbb{C}1$.

$(\mathrm{Irr}\,\mathcal{C} = \{\mathbb{1}\}.)$ $\quad H \in \mathcal{C}$ simple $\Leftrightarrow H = \mathbb{C}$.

Ex. 1.3

G : cpct (quantum) grp.

$\underset{\mathcal{C}}{\overset{}{\mathrm{Rep}\,G}} \overset{}{=} \{ v = (\pi_v, H_v) \}$ $\quad$ ($\mathrm{Rep}\,G \,\shortparallel\, \mathcal{C}$)

$\pi_v : G \longrightarrow B(H_v)$ fin dim unitary repn.

$\mathcal{C}(v, V) := \{ T \in B(H_v, H_V) \mid T\pi_v(t) = \pi_V(t) T \ \forall t \}$.

$v \otimes V := (\pi_v(t) \otimes \pi_V(t),\ H_v \otimes H_V)$

$v \in \mathcal{C}$ simple $\Leftrightarrow v$ irreducible.

Ex. 1.4

$$\begin{tikzcd}
\mathcal{C} \arrow[d, phantom, "\shortparallel"] & & \mathcal{D} \arrow[d, phantom, "\shortparallel"] \\
\mathrm{Rep}\,G \arrow[rr, "F"] & & \mathrm{Hilb}_f \\
v \arrow[u, phantom, "\in" description] \arrow[rr, mapsto] & & H_v \arrow[u, phantom, "\in" description]
\end{tikzcd}$$

forgetful functor

$$\begin{tikzcd}
F : \mathcal{C}(v, V) \arrow[r] & B(H_v, H_V) \\
T \arrow[u, phantom, "\in" description] \arrow[r, mapsto] & T \arrow[u, phantom, "\in" description]
\end{tikzcd}$$

injective map
⇓
faithful functor
(not surjective)

★ If $\mathcal{C} \xrightarrow{F} \mathrm{Hilb}_f$ unitary tensor functor, **faithful**

then $\exists ! G$ c.q.g.

$\exists E : \mathcal{C} \to \mathrm{Rep}\,G$ unitary tensor equiv.

s.t.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "E"'] & \mathrm{Hilb}_f \\
\mathrm{Rep}\,G \arrow[ur, "\mathrm{Forget}"'] &
\end{tikzcd}$$

~~mon.~~ naturally equiv.

(Woronowicz)

$$\begin{tikzcd}
& \mathcal{C} \arrow[dl, "F_1"'] \arrow[d, "F_2"] \arrow[dr, "F_3"] & \\
\mathrm{Hilb}_f & \mathrm{Hilb}_f & \mathrm{Hilb}_f \\
G_1 & G_2 & G_3
\end{tikzcd}$$

C^*-tensor cat is in a <u>higher</u> <u>level</u> than <u>c.q.gs</u>

Defn 1.5 (Rigid C^*-tensor cats)

$\mathcal{C}$: C^*-$\otimes$ cat is <u>rigid</u>

$\overset{\text{defn}}{\iff}$ $\forall X \in \mathcal{C}$ has a conj obj

i.e. $\exists Y \in \mathcal{C}$ $\exists S : \mathbb{1} \to Y \otimes X$

$\exists \bar{S} : \mathbb{1} \to X \otimes Y$

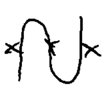
$\exists c > 0$

s.t.

conj. eq. →

- $S^* S = 1 = \bar{S}^* \bar{S}$
- $(S^* \otimes 1_Y)(1_Y \otimes \bar{S}) = \frac{1_Y}{c}$.

- $(\bar{S}^* \otimes 1_X)(1_X \otimes S) = \frac{1_X}{c}$



→ $c \geq 1$.

★ conj obj of X is unique up to unitary equiv.

We fix $\bar{X}$.

★ a rigid C^*-$\otimes$ cat has <u>dimension</u>

$X \in \mathcal{C}$

$d_{\mathcal{C}}(X) := \inf \{ c \mid (S, \bar{S}) \text{ solution of conj eq} \}$

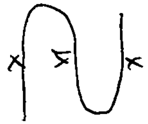
$= \min \{ c \mid \text{---}\text{''}\text{---} \}$

Fix $(R_X, \bar{R}_X)$ which attains the min.

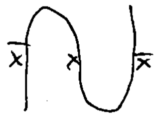
i.e.

$R_X : \mathbb{1} \to \bar{X} \otimes X$

$\bar{R}_X : \mathbb{1} \to X \otimes \bar{X}$.

- $R_X^* R_X = 1 = \bar{R}_X^* \bar{R}_X$
- 

$= \frac{1_X}{d(X)}$



$= \frac{1_{\bar{X}}}{d(X)}$.

$d_{\mathcal{C}}(X)$ intrinsic dim of X

We simply write $d(X)$ for $d_{\mathcal{C}}(X)$.

Properties of $d(X)$

- $d(\mathbb{1}) = 1$
- $d(\overline{X}) = d(X) \geq 1$.
- $d(X \oplus Y) = d(X) + d(Y)$
- $d(X \otimes Y) = d(X)\, d(Y)$

Traces on $\mathrm{End}(X)$

$$\begin{tikzcd} \mathrm{Tr}_X : \mathrm{End}(X) \arrow[r] & \mathbb{C} \\ T \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & R_X^*(1_{\overline{X}} \otimes T)R_X \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$$\qquad\qquad = \overline{R}_X^*(T \otimes 1_{\overline{X}})\overline{R}_X$$

Known that
$\forall S, T : X \to Y$.

$$d(X)\, \mathrm{Tr}_X(S^*T) = \mathrm{Tr}_Y(TS^*)\, d(Y)$$

$\rightsquigarrow \mathrm{Tr}_X$ is tracial.

Actually $\dim \underset{\mathrm{End}(X)}{\cancel{\mathcal{C}(X}} < +\infty$.

f.d. C^*-alg.

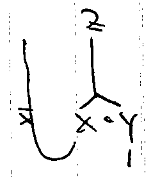
$\longrightarrow$ every X decomposes into simples Y_i

$X = Y_1 \oplus \cdots \oplus Y_m$.

Frobenius reciprocity

$^\forall X, Y, Z \in \mathcal{C}$

$$\begin{tikzcd} \mathcal{C}(X \otimes Y, Z) \arrow[r] & \mathcal{C}(Y, \overline{X} \otimes Z) \\ S \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & (1_{\overline{X}} \otimes S)(R_X \otimes 1_Y) \arrow[u, phantom, "\in" rotate=90] \\ d(X)(\overline{R}_X^* \otimes 1_Z)(1_X \otimes T) & T \arrow[l, mapsto] \end{tikzcd}$$



inverse map.

Similarly

$\mathcal{C}(X \otimes Y, Z) \longrightarrow \mathcal{C}(X, \cancel{Z}\, Z \otimes \overline{Y})$ defined

$$N_{X,Y}^{Z} := \dim \mathcal{C}(X\cdot Y, Z)$$

$$= \dim \mathcal{C}(Z, X\cdot Y)$$

$$N_{X,Y}^{Z} = N_{\bar{X},Z}^{Y} = N_{Z,\bar{Y}}^{X}$$

For $X, Y \in \mathcal{C}$ <u>simples</u>

$\mathcal{C}(X,Y) = 0$ or 1-dim.

[(∵) If $\underset{\neq 0}{a} \in \mathcal{C}(X,Y)$ $a^*a \in \mathcal{C}(X,X) = \mathbb{C}1$

$aa^* \in \mathcal{C}(Y,Y) = \mathbb{C}1$

a, b a invertible

$a^{-1}b \in \mathbb{C}$. $\rightarrow X \simeq Y$]

$$\mathcal{C}(1\!\!1, \bar{X}\otimes Y) \simeq \mathcal{C}(X,Y)$$

$$\mathcal{C}(1\!\!1, Y\cdot \bar{X}) \simeq \mathcal{C}(X,Y)$$

Hence. $1\!\!1 < \bar{X}\cdot Y \Leftrightarrow X \cong Y \Leftrightarrow 1\!\!1 < Y\cdot\bar{X}$.

- Let X. ~~Y. Z~~ $\in \mathcal{C}$ simples
  & $Y \in \mathcal{C}$.

$\rightarrow$ $\mathcal{C}(X,Y)$ Hilb sp by

$\langle S,T\rangle 1_X = T^*S$

$ONB(X,Y)$: an ONB of $\mathcal{C}(X,Y)$

- $X, Y, Z \in \mathcal{C}$ simples.

$$\begin{tikzcd} \mathcal{C}(X, Y\cdot Z) \arrow[r, "F_\ell"] & \mathcal{C}(Z, \bar{Y}\cdot X) \\ S \arrow[u, phantom, "\in"] \arrow[r, mapsto] & T \arrow[u, phantom, "\in"] \end{tikzcd}$$

$$\frac{d(X)^{\frac{1}{2}}}{d(Y)^{\frac{1}{2}}d(Z)^{\frac{1}{2}}}T := (1_{\bar{Y}}\cdot S^*)(R_Y\cdot 1_Z)$$

$$\begin{tikzcd} \mathcal{C}(X, Y\cdot Z) \arrow[r, "F_r"] & \mathcal{C}(Y, X\cdot \bar{Z}) \\ S \arrow[u, phantom, "\in"] \arrow[r, mapsto] & T \arrow[u, phantom, "\in"] \end{tikzcd}$$

$$\frac{d(X)^{\frac{1}{2}}}{d(Y)^{\frac{1}{2}}d(Z)^{\frac{1}{2}}}T := (S^*\cdot 1_{\bar{Z}})(1_Y\cdot \bar{R}_Z)$$

unitaries

In particular $\underset{F_r}{F_\ell} : ONB \rightarrow ONB$

Ex. 1.6

M : infinite factor.

$\mathcal{C} := \mathrm{End}(M)_0$

obj : $\rho \in \mathrm{End}(M)_0$

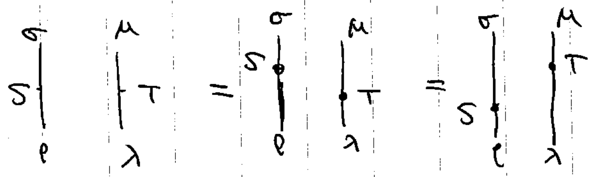
morp : $(\rho . \sigma)$

$\otimes$: $\rho \otimes \sigma := \rho\sigma \qquad 1 = \mathrm{id}_M$

$S \otimes T := S\rho(T) = \sigma(T)S$

$S : \rho \to \sigma$

$T : \lambda \to \mu$



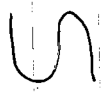
$\to$ $\mathcal{C}$ rigid C^*-tensor category

The meaning of $d(\rho)$.

Let $(S_\rho . \bar{S}_\rho)$ sol. of conj eq $(\rho . \bar{\rho})$

$S_\rho : 1 \to \bar{\rho}\otimes\rho$

$\bar{S}_\rho : 1 \to \rho\otimes\bar{\rho}$


$= \frac{1}{d_s(\rho)}$

Put $\phi_\rho^s : M \to M$

$x \mapsto S_\rho^* \bar{\rho}(x) S_\rho$ u.c.p. normal faithful

$$\left[\begin{array}{l} (\because) \; \phi_\rho(x^*x) = 0 \Rightarrow \bar{\rho}(x) S_\rho = 0 \\ \rightsquigarrow \rho\bar{\rho}(x)\rho(S_\rho) = 0 \\ \rightsquigarrow \bar{S}_\rho^* \rho\bar{\rho}(x)\rho(S_\rho) = 0 \\ \qquad \| \\ x \; \bar{S}_\rho^* \rho(S_\rho) = \frac{1}{d_s(\rho)} x \end{array} \right]$$

(* ~~Easy to see ϕ_ρ^s does not dep~~)

$\phi_\rho^S \cdot \rho = \mathrm{id}_M$ left inverse.

$\to E_\rho^S := \rho \circ \phi_\rho^S : M \to \rho(M)$ cond exp.

Let e^{s}_{ρ} Jones proj w.r.t. E^{s}_{ρ}

i.e. $\rho(M) < M \subset \langle M. e^{s}_{\rho} \rangle$

$$M e^{s}_{\rho} M = \langle M. e^{s}_{\rho} \rangle$$

$$e^{s}_{\rho} x e^{s}_{\rho} = E^{s}_{\rho}(x) e^{s}_{\rho} \quad \forall x \in M$$

Set $m^{s} := d^{s}(\rho) \bar{S}^{*}_{\rho}$ $\nwarrow$ $d^{s}(\rho)$ denotes 

Then we can check

$$m^{s} e^{s}_{\rho} m^{s*} = 1. \in \langle M. e^{s}_{\rho} \rangle$$

Index theory (Jones, Pimsner-Popa, Kosaki, Watatani)

$\rightsquigarrow$

$$\mathrm{Ind} E^{s}_{\rho} := m^{s} m^{s*} = d^{s}(\rho)^{2}.$$

$$\geqq d_{e}(\rho)^{2} = \mathrm{Ind} E_{\rho}$$

$\nearrow$ intrinsic dim.

$(R_{\rho}, \bar{R}_{\rho})$ std sol.

indep of $(\bar{\rho}, R_{\rho}\bar{R}_{\rho})$ $\longrightarrow$ $\phi_{\rho}(x) := R^{*}_{\rho} \bar{\rho}(x) R_{\rho}$

$$E_{\rho} := \rho \circ \phi_{\rho}$$

Known that (USE $[DE = DE_{\rho}]_{t}$)

$\forall E: M \to \rho(M)$ f.n. cond.exp.

$\exists !$ $h \in \rho(M)' \cap M$ positive inv.

s.t.

$$E(x) = E_{\rho}(h x h)$$

$$= \rho(R^{*}_{\rho} \bar{\rho}(h x h) R_{\rho})$$

~~$= \rho(R_{\rho}$~~

$\rightarrow \exists c > 0$ $S_{\rho} := \bar{\rho}(h) R_{\rho}$ $\nearrow$ $E = E^{s}$

$\bar{S}_{\rho} := \underset{\vee}{c}\,h^{-1} \bar{R}_{\rho}$ ~~std~~ sol. of conj. eq.

0

$\longrightarrow$ $\bar{S}^{*}_{\rho} \bar{S}_{\rho} = 1$

$\shortparallel$

$c^{2} \bar{R}^{*}_{\rho} h^{-2} \bar{R}_{\rho}$

$(\bar{R}^{*}_{\rho} h^{-2} \bar{R}_{\rho})^{\frac{1}{2}} (\bar{R}^{*}_{\rho} h^{2} \bar{R}_{\rho})^{\frac{1}{2}} \geqq \bar{R}^{*}_{\rho} 1 \bar{R}_{\rho} = 1$

$\shortparallel$

$\mathrm{Tr}_{\rho}(h^{2})$

$\shortparallel$

"$R^{*}_{\rho} (1 \otimes h^{2}) R_{\rho}$"

$\shortparallel$

$R^{*}_{\rho} \bar{\rho}(h^{2}) R_{\rho} = S^{*}_{\rho} S_{\rho} = 1$

$\rightarrow c \leqq 1.$

$$S_\rho^* \bar{\rho}(\bar{S}_\rho) = \frac{c}{d(\rho)} = \frac{1}{\enclose{circle}{\frac{d(\rho)}{c}}} =: ds(\rho)$$

$$\mathrm{Ind}\, E = \mathrm{Ind}\, E^S = d_S(\rho)^2 \geqq \mathrm{Ind}\, E_\rho$$

Hence $E_\rho : M \to \rho(M)$ minimal exp (Hiai)

$$\mathrm{Ind}\, E_\rho = d(\rho)^2$$

$\|$ often denoted by

$[M : \rho(M)]_0$.

Amenability of $\mathcal{C}$

$\mathrm{Irr}\,\mathcal{C}$. σ: measure

$\sigma(x) = d(x)^2$. $x \in \mathrm{Irr}\,\mathcal{C}$

$\mathcal{F} \subset \mathrm{Irr}\,\mathcal{C}$

$$|\mathcal{F}|_\sigma := \sum_{x \in \mathcal{F}} d(x)^2$$

Defn. 1.7

- $F . K \subset\subset \mathrm{Irr}\,\mathcal{C}$. $\delta > 0$.

  K is $(F . \delta)$-inv.

  $\overset{\text{defn}}{\Longleftrightarrow}$ $|\mathcal{F} \cdot K \,\triangle\, K|_\sigma < \delta |K|_\sigma$.

  where

  $\cdot$ $\mathcal{F} \cdot K := \{ z \in \mathrm{Irr}\,\mathcal{C} \mid {}^{\exists} x \in \mathcal{F}$
  $\qquad y \in K$
  $\qquad z < x \otimes y$.

  $\cdot$ $\triangle$: symmetric difference.

- $\mathcal{C}$ amenable $\overset{\text{defn}}{\Longleftrightarrow}$ ${}^{\forall}\mathcal{F} \subset\subset \mathrm{Irr}\,\mathcal{C}$ ${}^{\forall}\delta > 0$

  ${}^{\exists} K \subset\subset \mathrm{Irr}\,\mathcal{C}$ $(F\,\delta)$-inv.

Rem

${}^{\exists}$ many equiv conditions to amenability.

We set

$$P_x(y. z) := \frac{d(z)}{d(x)\,d(y)} N_{x.y}^{z}$$

for $x . y . z \in \mathrm{Irr}\,\mathcal{C}$.

(Recall $N_{x.y}^{z} = \dim \mathcal{C}(z,\ x \otimes y)$.)

The Transition prob.

$$\begin{tikzcd} x \arrow[r, bend left] & y \arrow[r] \arrow[dr] & z_1 \\ & & z_2 \end{tikzcd}$$

left multiplication

$$\sum_z P_x(y.z) = 1.$$

Lem. $K : (F.\delta)-\mathrm{inv}.$

$$\sum_{(x.y.z)\in F\times K\times K^c} d(x)^2 d(y)^2 p_x(y.z) \qquad (*)$$

$$= \sum_{(x.y.z)\in F\times K^c\times K} d(x)^2 d(y)^2 p_{x}(y.z)$$

$$\leqq \delta\, |F|_\sigma\, |K|_\sigma$$

Proof.

$$p_x(y.z) = \frac{d(z)}{d(x)d(y)} N_{x,y}^{z} \underset{\uparrow}{=} 0$$

$x\in F \quad y\in K \qquad$ if $z \notin F.K.$

$$(*) = \sum_{(x.y.z)\in F\times K\times F\cdot K\setminus K^{\varepsilon}} d(x)^2 d(y)^2 p_x(y.z)$$

$$\leqq \sum_{(x,y.z)\in F\times \mathrm{Irre}\times F\cdot K\setminus K^{\varepsilon}} d(x)d(y)\,d(z)\,N_{x,y}^{z} \qquad \begin{array}{c} d(y)^2 p_x(y.z) \\ \shortparallel \\ d(z)^2 p_x(\cancel{y.z})\\ z,y \end{array}$$

$$= \sum_{(x.z)\in \text{—''—}} d(x)^2 d(z)^2$$

$$= |F|_\sigma\, |F\cdot K\setminus K^{\varepsilon}|_\sigma \leqq |F|_\sigma\, \delta\, |K|_\sigma. \quad \boxed{}$$

Section 2 Cocycle actions

§2.1. Cocycle actions.

Defn 2.1

$(\alpha, c) : \mathcal{C} \curvearrowright M$ cocycle action

$\overset{\text{defn}}{\Longleftrightarrow}$ $(\alpha, c) : \mathcal{C} \longrightarrow \mathrm{End}(M)_0$ unitary tensor functor.

i.e.

$\forall X \in \mathcal{C}$. $\alpha_X \in \mathrm{End}(M)_0$ assigned.

$\forall X, Y \in \mathcal{C}$ $c_{X,Y} \in M$ unitary

$\forall T \in \mathcal{C}(X,Y)$ $\alpha(T) \in M$

$\quad\quad \|$ denoted by T^α or $[T]^\alpha$

with

- $\alpha_{\mathbb{1}} = \mathrm{id}_M$
- $c_{X,\mathbb{1}} = 1 = c_{\mathbb{1},X}$
- $\alpha_X \cdot \alpha_Y = \mathrm{Ad}\, c_{X,Y} \cdot \alpha_{X \otimes Y}$
- $c_{X,Y}\, c_{X\otimes Y, Z} = \alpha_X(c_{Y,Z})\, c_{X, Y\otimes Z}$
- $T^\alpha \alpha_X(x) = \alpha_Y(x) T^\alpha$ $\forall x \in M$ $T \in \mathcal{C}(X,Y)$
- $[ST]^\alpha = [S]^\alpha T^\alpha$

  $[S^*]^\alpha = (S^\alpha)^*$ unitarity

  $1_X^\alpha = 1$ $(1_X \in \mathrm{End}(X))$

- $S : X \to Y$, $T : Z \to W$

  $c_{Y,W}\, [S \otimes T]^\alpha = S^\alpha \alpha_X(T^\alpha)\, c_{X,Z}$

  naturality

☆ $T \mapsto T^\alpha$ faithful.

☆ If $c_{X,Y} = 1$ $\forall X, Y \in \mathcal{C}$. $(\alpha, 1)$ is called an action.

☆ (unitary perturbation)

If $v_X \in M$ unitary $(X \in \mathcal{C})$ given,

Put

$$\alpha^v_X := \mathrm{Ad}\, v_X \cdot \alpha_X$$

$$c^v_{X,Y} := v_X\, \alpha_X(v_Y)\, c_{X,Y}\, v_{X\otimes Y}^*$$

$$T^{\alpha^v} := v_Y T^\alpha v_X \quad T \in \mathcal{C}(X,Y)$$

$\rightsquigarrow$ $(\alpha^v, c^v) : \mathcal{C} \curvearrowright M$ cocycle action.

Left inverse of α_X

$(\alpha, c) : \mathcal{C} \curvearrowright M$.

Set $\phi^\alpha_X : M \to M$

$$\phi^\alpha_X(x) = R_X^{\alpha *} C^*_{\bar{X},X}\, \alpha_{\bar{X}}(x)\, C_{\bar{X},X}\, R^\alpha_X$$

for $x \in M$.

Lem. 2.2

(1) ϕ^α_X is faithful normal ucp.

(2) $\phi^\alpha_X \cdot \alpha_X = \mathrm{id}_M$

(3) $\phi^\alpha_{X\circ Y}\, \mathrm{Ad}C_{X,Y} = \phi^\alpha_Y \cdot \phi^\alpha_X$

Proof.

(1) Suppose $x \geq 0$ with $\phi^\alpha_X(x) = 0$

$\rightsquigarrow \alpha_{\bar{X}}(x)\, C_{\bar{X},X}\, R^\alpha_X = 0$

$$\alpha_X(\alpha_{\bar{X}}(x))\, \alpha_X(C_{\bar{X},X})\, \alpha_X(R^\alpha_X) = 0$$

$$\shortparallel$$

$$\cancel{C_{X,\bar{X}}}\, \alpha_{X\circ\bar{X}}(x)\, C^*_{X,\bar{X}}\, \alpha_X(C_{\bar{X},X})\, \alpha_X(R^\alpha_X)$$

$\rightsquigarrow \bar{R}_X^{\alpha *}\, \alpha_{X\circ\bar{X}}(x)\, C^*_{X,\bar{X}}\, \alpha_X(C_{\bar{X},X})\, \alpha_X(R^\alpha_X) = 0$

$\shortparallel$ 2-cocycle rel.

$$x\, \bar{R}_X^{\alpha *} \cdot C_{X\circ\bar{X},X}\, C^*_{X,\bar{X}\circ X}\, \alpha_X(R^\alpha_X)$$

$\shortparallel$ naturality

$$x\, \cancel{\bar{R}}\, C_{1,X}[\bar{R}_X^{\alpha *} \otimes 1_X]^\alpha\, [1_X \circ R_X]^\alpha\, C^*_{X,1}$$

$\shortparallel$ functoriality

$$x\, [(\bar{R}_X^* \cdot 1_X)(1_X \circ R_X)]^\alpha$$

$$\shortparallel$$

$$x\, [d(X)^{-1} 1_X]^\alpha$$

$$\shortparallel$$

$$d(X)^{-1} x$$

(2)
$$\begin{aligned}
\phi^\alpha_X(\alpha_X(x)) &= R_X^{\alpha *} C^*_{\bar{X},X}\, \alpha_{\bar{X}}(\alpha_X(x))\, C_{\bar{X},X}\, R^\alpha_X \\
&= R_X^{\alpha *}\, \alpha_{\bar{X}\circ X}(x)\, R^\alpha_X \\
&= x\, R_X^{\alpha *} R^\alpha_X \\
&= x.
\end{aligned}$$

(3) $\bar{Y}\circ\bar{X}$, $X \circ Y$. conj. pair

$(1_{\bar{Y}} \circ R_X \cdot 1_Y)\cdot R_Y$ & $(1_X \cdot \bar{R}_Y \cdot 1_{\bar{X}})\cdot \bar{R}_X$

std. sol. of conj eq.

$$\phi^{\alpha}_{X\otimes Y}(c_{X,Y}^{*}\, x\, c_{X,Y}^{\overline{*}})$$

$$= [R_{\bar{Y}}^{*}(1_{\bar{Y}}\cdot R_X^{*}\cdot 1_Y)]^{\alpha}\, c_{\bar{Y}\otimes\bar{X},X\cdot Y}^{*}$$

$$\alpha_{\bar{Y}\otimes\bar{X}}(c_{X,Y}^{*}\, x\, c_{X,Y}^{\overline{*}})$$

$$c_{\bar{Y}\cdot\bar{X},X\cdot Y}\quad[(1_{\bar{Y}}\cdot R_X\cdot 1_Y)R_Y]^{\alpha}$$

$$= R_Y^{\alpha *}[1_{\bar{Y}}\cdot R_X^{*}\cdot 1_Y]^{\alpha}\, c_{\bar{Y}\otimes\bar{X},X\cdot Y}^{*}$$

$$c_{\bar{Y},\bar{X}}^{*}\ \alpha_{\bar{Y}}(\alpha_{\bar{X}}(c_{X,Y}^{*}\, x\, c_{X,Y}^{\overline{*}}))\, c_{\bar{Y},\bar{X}}^{\overline{*}}$$

$$c_{\bar{Y}\cdot\bar{X},X\cdot Y}\ \ldots$$

$$= R_Y^{\alpha *}[1_{\bar{Y}}\cdot R_X^{*}\cdot 1_Y]^{\alpha}\ c_{\bar{Y},\bar{X}\cdot X\cdot Y}^{*}\ \alpha_{\bar{Y}}(c_{\bar{X},X\cdot Y}^{*})$$

$$\alpha_{\bar{Y}}(\alpha_{\bar{X}}(c_{X,Y}^{*}\, x\, c_{X,Y}^{*}))$$

$$\ldots$$

$$= R_Y^{\alpha *}\, c_{\bar{Y},Y}^{*}\ \alpha_{\bar{Y}}([R_X^{*}\cdot 1_Y]^{\alpha})$$

$$\alpha_{\bar{Y}}(\underbrace{c_{\bar{X},X\cdot Y}^{*}\ \alpha_{\bar{X}}(c_{X,Y}^{*}}_{=}\, x\, c_{X,Y}^{\overline{*}})c_{\bar{X},X\cdot Y})$$

$$\ldots\qquad \alpha_{\bar{Y}}(c_{\bar{X}\cdot X,Y}^{*}\ c_{\bar{X},X}^{*}\ \alpha_{\bar{X}}(x)\ \ldots\qquad)$$

$$= R_Y^{\alpha *}\, c_{\bar{Y},Y}^{*}\ \alpha_{\bar{Y}}(R_X^{\alpha *}\, c_{\bar{X},X}\, \alpha_{\bar{X}}(x)\ \ldots$$

$$= \phi_Y^{\alpha}\,\phi_X^{\alpha}(x).$$

☆ ϕ_X^{α} does not dep on $\bar{X}$, & $(R_X, \bar{R}_X)$.

Lem. 23 (Decomposition rule)

$X = X_1 \oplus \cdots \oplus X_n$

i.e. $S_k : X_k \to X$ isometry

& $\sum_k S_k S_k^{*} = 1_X$

Then

$$d(X)\,\phi_X^{\alpha}(x) = \sum_{k=1}^{n} d(X_k)\,\phi_{X_k}^{\alpha}(S_k^{\alpha}\, x\, S_k^{\alpha *})$$

$\forall x \in M$

Proof.

$\bar{X} := \bar{X}_1 \oplus \cdots \oplus \bar{X}_n$ $\quad T_k : \bar{X}_k \xrightarrow[\text{isom.}]{} \bar{X}$

Known that

$\sum_k T_k T_k^{*} = 1_{\bar{X}}$.

Use.

$$R_X := \sum_{k} (T_k \otimes S_k) \cdot R_{X_k} \frac{d(X_k)^{\frac{1}{2}}}{d(X)^{\frac{1}{2}}}$$

$$\bar{R}_X := \sum_{k} (S_k \otimes T_k) \cdot \bar{R}_{X_k} \frac{d(X_k)^{\frac{1}{2}}}{d(X)^{\frac{1}{2}}}$$

▨

Lem 2.4.

$$\alpha_X(\varphi) := \varphi \circ \phi_X^\alpha \qquad \varphi \in M_* \quad X \in \mathcal{C}$$

$$\alpha_X : M_* \to M_* \qquad \text{isometric linear.}$$

Lem 2.4.

(1) $\alpha_X(a\varphi b) = \alpha_X(a)\,\alpha_X(\varphi)\,\alpha_X(b)$

(2) $\alpha_X(\alpha_Y(\varphi)) = C_{X,Y}\,\alpha_{X\cdot Y}(\varphi)\,C_{X,Y}^*$

(3) $^\forall S : X \to Y$

$$d(X)\, S\, \alpha_X(\varphi) = d(Y)\, \alpha_Y(\varphi)\, S.$$

Proof

(1) Direct.

(2) $\alpha_X(\alpha_Y(\varphi)) = \varphi\, \phi_Y^\alpha \cdot \phi_X^\alpha$

$= \varphi\, \phi_{X\cdot Y}^\alpha \cdot \mathrm{Ad}\, C_{X,Y}^*$

$= C_{X,Y}\,\alpha_{X\cdot Y}(\varphi)\, C_{X,Y}^*$

(3) Use irred. decomposition. of X & Y.

& decomposition rules of ϕ_X^α.

▨

§2.2. Freeness of (α, C)

Defn 2.5

$(\alpha, C) : \mathcal{C} \curvearrowright M$ is free

$\overset{\text{defn}}{\Longleftrightarrow}$ $(\alpha, C) : \mathcal{C} \to \mathrm{End}(M)_0$ "full"

i.e.

$$(\alpha_X, \alpha_Y) = \alpha(\mathcal{C}(X,Y))\, Z(M) . \quad (*)$$

$\forall X, Y \in \mathcal{C}$.

$$= \mathrm{span}\{ T^\alpha z \mid T \in \mathcal{C}(X,Y),\ z \in Z(M) \}$$

Lem 2.6

(α, C) free

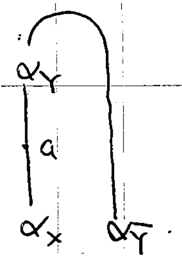
$\Longleftrightarrow (\alpha_X, \alpha_{\mathbb{1}}) = \{0\} \quad \forall X \in \mathrm{Irr}\,\mathcal{C} \setminus \{\mathbb{1}\}$

Proof.

$\Longrightarrow$ ok

$\Longleftarrow$ We show $(*)$. WMA $X, Y \in \mathrm{Irr}\,\mathcal{C}$.

$a \in (\alpha_X, \alpha_Y)$



$$b := \bar{R}_Y^{\alpha *} C_{Y,\overline{Y}}^{*}\, a\, C_{X,\overline{Y}}$$

$$\in (\alpha_{X \otimes \overline{Y}}, \alpha_{\mathbb{1}})$$

$$b = \sum_{Z,T} \boxed{b T^\alpha} T^{\alpha *} \qquad Z \in \mathrm{Irr}\,\mathcal{C}$$

$$T \in \mathrm{ONB}(Z, X \cdot \overline{Y}).$$

$$\Uparrow$$

$$(\alpha_Z, \alpha_{\mathbb{1}}) = 0 \text{ if } Z \neq \mathbb{1}$$

$$= \sum_T b T^\alpha T^{\alpha *} \qquad T \in \mathrm{ONB}(\mathbb{1}, X \cdot \overline{Y})$$

$\Uparrow$ one dim or zero.

$$= \boxed{b \bar{R}_X^\alpha} \bar{R}_X^{\alpha *} \delta_{X,Y}$$

$$\overset{\shortparallel}{c} \ \cancel{Z} \ \overset{\Uparrow}{(\alpha_{\mathbb{1}}, \alpha_{\mathbb{1}})} = Z(M)$$

$$\to \bar{R}_Y^{\alpha *} C_{Y,\overline{Y}}^{*}\, a = \delta_{X,Y} \overset{c}{\cancel{z}}\, \bar{R}_X^{\alpha *} C_{X,\overline{Y}}^{*}.$$

$$\curvearrowleft \alpha_X(C_{\overline{Y},Y} R_Y^\alpha)$$

$$d(Y)^{-1} a = \delta_{X,Y}\, c\, \underbrace{\bar{R}_X^{\alpha *} C_{X,\overline{Y}}^{*} \alpha_X(C_{\overline{Y},Y} R_Y^\alpha)}_{\shortparallel\ \delta_{X,Y}\, d(X)^{-1}}.$$

$$a = c \cdot \delta_{X,Y}$$

■

__Lem.__ 8.7

$(\alpha, c) : \mathcal{C} \curvearrowright M$ free

(1) $\alpha_X \restriction Z(M) \in \mathrm{Aut}(Z(M))$
$X \in \mathrm{Irr}\,\mathcal{C}$

(2) $\alpha_X \alpha_Y = \alpha_Z$ on $Z(M)$ if $X, Y, Z \in \mathrm{Irr}\,\mathcal{C}$
$Z \prec X \otimes Y$.

__proof__

(1) $(\alpha_X, \alpha_X) = \alpha(\mathcal{C}(X,X))\, Z(M)$

$= Z(M)$

$$\alpha_X(M)' \cap M = Z(M)$$
$$\cup$$
$$\alpha_X(Z(M)) \qquad \forall X \in \mathrm{Irr}\,\mathcal{C}$$

Let $x \in Z(M)$. $y := \alpha_{\bar X}(x) \in \alpha_{\bar X}(Z(M)) \underset{Z(M)}{\cap}$

$$x = \phi^{\alpha}_{\bar X}(y) = \bar R^{\alpha *}_X C^*_{X,\bar X}\, \underset{\in}{\overset{Z(M)}{}} \alpha_X(y) \cdots$$

$= \alpha_X(y)$ ∎

(2) $x \in Z(M)$

$$\alpha_X(\underset{\underset{Z(M)}{\cap}}{\alpha_Y(x)}) = C_{X,Y}\, \alpha_{X\cdot Y}(x)\, C^*_{X,Y}$$

$$\alpha_X(\alpha_Y(x)) = \alpha_{X\cdot Y}(x)$$

$S^* \curvearrowright \quad \curvearrowleft S \qquad S : Z \to X \cdot Y$.

∎

§2.3 Central Freeness

$(\alpha, c) : \mathcal{C} \curvearrowright M$ cocycle action

Recall M^{ω} & M_{ω}

$(x_n)^{\omega}$ bdd. multiplier of J^{ω} ↗ M^ω ; $(x_n)^{\omega}$ central ↗ M_ω

Set $(\alpha^{\omega}, c) : \mathcal{C} \curvearrowright M^{\omega}$ by

$$\alpha_X^{\omega}((x_n)^{\omega}) := (\alpha_X(x_n))^{\omega}$$

(∵) Well- ~~define~~d ?

$(x_n) \in \mathcal{M}^{\omega}_{\cancel{*}}$, $(y_n) \in J^{\omega}$ trivial seq.

$$\| y_n \alpha_X(x_n) \|^2_{\alpha_X(\varphi)}$$

$$= \alpha_X(\varphi)(\alpha_X(x_n^*) y_n^* y_n \alpha_X(x_n))$$

$$= \varphi(x_n^* \phi_X^{\alpha}(y_n^* y_n) x_n)$$

$$= \| \underbrace{\phi_X^{\alpha}(y_n^* y_n)^{\frac{1}{2}}}_{\text{converges to } 0} x_n \|_{\varphi} \longrightarrow 0 .$$

We will simply write α_X for α_X^{ω} $\longrightarrow \alpha_X^{\omega}(\varphi^{\omega}) = \alpha_X(\varphi)^{\omega}$

ϕ_X^{α} for $\phi_X^{\alpha^{\omega}}$

✳ We use the same 2-cocycle $c_{X,Y} \in M$ & $T^{\alpha} \in M$ $T \in \mathcal{C}(X, Y)$.

✳ $\alpha_X(M_\omega) \not\subseteq M_\omega$ in general.

$\phi_X^{\alpha}(M_\omega) \subseteq M_\omega$

(∵)

$x = (x_n)^{\omega} \in M_\omega$

$$\phi_X^{\alpha}(a)\varphi^{\omega} = [x\, \alpha_X(\varphi^{\omega})] \circ \alpha_X$$

$$= [\alpha_X(\varphi^{\omega})\, x] \circ \alpha_X$$

$$= \varphi^{\omega} \cdot \phi_X^{\alpha}(x)$$

$$\phi_X^{\alpha}(a)\varphi(x)$$

$$= \varphi(x \phi_X^{\alpha}(a))$$

$$= \varphi(\phi_X^{\alpha}(\alpha_X(x) a))$$

$$= \alpha_X(\varphi)(\alpha_X(x) a)$$

$$= a\, \alpha_X(\varphi)(\alpha_X(x)) .$$

皿

Defn 2.8

$(\alpha, c) : \mathcal{C} \curvearrowright M$ is

- **centrally free** $\overset{\text{defn}}{\Leftrightarrow}$ $\forall X \in \operatorname{Irr}\mathcal{C} \setminus \{\mathbb{1}\}$
  $\nexists \underset{\neq 0}{a} \in \underset{\sim}{M}$ s.t. $xa = \alpha_X(x) a \quad \forall x \in M_\omega$
- **strongly free** $\overset{\text{defn}}{\Longleftrightarrow}$ $\forall X \in \operatorname{Irr}\mathcal{C} \setminus \{\mathbb{1}\}$
  $\forall Q \subset M^\omega$ countably gen. vN.
  $\nexists \underset{\neq 0}{a} \in M^\omega$ s.t. $ax = \alpha_X(x) a \quad \forall x \in Q' \cap M_\omega$

Facts

- centrally free $\Leftrightarrow$ strongly free
- M factor
  centrally free $\Leftrightarrow$ Each α_X is centrally non-trivial $(X \in \operatorname{Irr}\mathcal{C} \setminus \{\mathbb{1}\})$
  i.e. $\alpha_X \neq \mathrm{id}_{M_\omega}$ on M_ω

Lem 2.9

$(\alpha, c) : \mathcal{C} \curvearrowright M$ centrally free

$\forall X, Y \in \mathcal{C} \quad \forall Q \subset M^\omega$ countably gen vN.

$$\{a \in M^\omega \mid a\,\alpha_X(x) = \alpha_Y(x)\, a \quad \forall x \in Q' \cap M_\omega\}$$

$$= \alpha(\mathcal{C}(X, Y))\, \alpha_X((Q' \cap M_\omega)' \cap M^\omega)$$

(∵) Similar proof to Lem 2.6 ∎

§9.4 Loc. quantization.

For $F \Subset \mathrm{Irr}\,\mathcal{C}\setminus\{\mathbb{1}\}$,

we construct $e \in M_\omega$ proj s.t.

$$\begin{tikzcd} e \arrow[r, bend left] \arrow[rr, bend left] & \alpha_{x_1}(e) & \alpha_{x_2}(e) \cdots \end{tikzcd}$$

orthogonal proj's. $x_1 \neq x_2 \neq \cdots$

Lem. 9.10

- $(\alpha, \mathcal{C}) : \mathcal{C} \curvearrowright M$ cent. free.
- $\varphi \in M_*$: faithful state

$\varphi|_{Z(M)}$: α-inv. $\varphi \cdot \alpha_x = \varphi$.

(NOTE: $(\alpha_x, \mathcal{C})$ preserves $Z(M)$)

$\forall F \Subset \mathrm{Irr}\,\mathcal{C}\setminus\{\mathbb{1}\}$. $\forall \delta > 0$.

$\exists n \in \mathbb{N}$ $\exists \{q_r\}_{r=0}^{n}$ part. of 1 in $Q' \cap M_\omega$

s.t.

(1) $|q_0|_{\varphi^\omega} < \delta$

(2) $\sum_{x \in F} |q_r \alpha_x(q_r)|_{\varphi^\omega} < \delta |q_r|_{\varphi^\omega}$ $r \geq 1$.

Proof.

$F = \{x_1, \cdots x_m\}$. $(x_i \neq x_j)$

$X := \underset{X_0}{\underset{\parallel}{\mathbb{1}}} \oplus X_1 \oplus \cdots \oplus X_m \in \mathcal{C}$ $\quad T_k : X_k \to T$.

Set

$N := \alpha_X(Q' \cap M_\omega) \subset M^\omega$.

$\rightsquigarrow$ $N \subset (M^\omega)_{\alpha_X(\varphi^\omega)}$

(∵) $x \in Q' \cap M_\omega$.

$\alpha_X(x)\,\alpha_X(\varphi^\omega) = \alpha_X(x\varphi^\omega) = \alpha_X(\varphi^\omega x) = \alpha_X(\varphi^\omega)\alpha_X(x)$ ∎

$\rightsquigarrow$ $\exists!\, E : M^\omega \to N \vee (N' \cap M^\omega)$ $\alpha_X(\varphi^\omega)$-preserving cond. exp.

(∵) $\sigma_t^{\alpha_X(\varphi^\omega)}(x) = x$ on N.

$\rightarrow$ $\sigma_t^{\alpha_X(\varphi^\omega)}$-inv. $N \vee (N' \cap M^\omega)$ is .

$\rightarrow$ Takesaki thm (JFA) ∎

By Lem 2.9

$$N\vee(N'\cap M^\omega) = \underset{\text{commuting}}{\alpha_X(\mathcal{C}(X,X))\ \alpha_X((Q'\cap M_\omega)\vee((Q'\cap M_\omega)'\cap M^\omega))}$$

$$\left((\because)\quad T: X\to X.\quad T^\alpha \alpha_X(y) = \alpha_X(y) T^\alpha \quad \forall y\in M^\omega \right)$$

$$\mathcal{C}(X,X) = \mathbb{C}T_1T_1^* + \cdots + \mathbb{C}T_mT_m^*.$$

$$\alpha(\mathcal{C}(X,X)) = \mathbb{C}T_1^\alpha T_1^{\alpha*} + \cdots + \mathbb{C}T_m^\alpha T_m^{\alpha*}$$

$$(\cong \mathbb{C}^m)$$

In particular $T_k^\alpha T_k^{\alpha *} \in \mathcal{Z}(N\vee(N'\cap M^\omega))$.

$$\rightsquigarrow\quad E(T_0^\alpha T_k^{\alpha*}) = 0 \quad k \overset{?}{\geq} 1 \quad \text{central}$$

$$\left((\because)\quad E(T_0^\alpha \overset{\shortparallel}{T_k^{\alpha*}} \underbrace{T_k^\alpha T_k^{\alpha*}}_{\substack{\Uparrow\\ N\vee(N'\cap M^\omega)}}) = E(T_0^\alpha T_k^{\alpha*}) T_k^\alpha T_k^{\alpha*} \right.$$

$$\left. = E(T_k^\alpha \underbrace{T_k^{\alpha*} T_0^\alpha}_{\delta_{k,0}} T_k^{\alpha*}) \right)$$

By Popa's local quantization. (CBMS)

$\exists\ \{q_r\}_{r\in J}$ fin. part of 1 in $Q'\cap M_\omega$ st.

$$\Big\| \sum_{r\in J} \alpha_X(q_r) T_0^\alpha T_k^{\alpha*} \alpha_X(q_r) \Big\|_{\alpha_X(\varphi^\omega)} < \varepsilon$$

$$\left[\begin{array}{l} \bigstar\ N\subset M_\varphi.\quad \forall x_1 \cdots x_m \in M \text{ st. } E_{N\vee(N'\cap M)}(x_k) = 0. \\ \quad \forall \varepsilon > 0 \\ \quad \exists q_1, \cdots q_t \in N \text{ proj.} \\ \quad \Big\| \sum_{r=1}^{t} q_r x_k q_r \Big\|_\varphi < \delta. \quad k = 1 \cdots m \end{array} \right]$$

$T_k: X_k \to X$ $\qquad T_0: \mathbb{1} \to X$

$$LHS^2 = \sum_{r\in J} \alpha_X(\varphi^\omega)\big(\alpha_X(q_r) T_k^\alpha T_0^{\alpha*}\ \underline{\alpha_X(q_r) T_0^\alpha T_k^{\alpha*} \alpha_X(q_r)}\big)$$

$$\shortparallel$$

$$(\quad |^2 \qquad \alpha^u \neq m \ldots\ \cancel{T_0^\alpha q_r T_k^{\alpha*}}$$

$$\alpha_X(q_r) T_0^\alpha\ \alpha_{X_k}(q_r) T_k^{\alpha*}$$

$$= \sum_{r\in J} \alpha_X(\varphi^\omega)(\cancel{T_k^\alpha}$$

$$\shortparallel$$

$$\cancel{\alpha}(q_r\, \alpha_{X_k}(q_r)).$$

$$T_k^\alpha\ |q_r \alpha_{X_k}(q_r)|^2\ T_k^{\alpha*}$$

$$= \sum_{r\in J} T_k^{\alpha*} \alpha_X(\varphi^\omega) T_k^\alpha (\ |\ |^2)$$

$$= \sum_k \frac{d(X_k)}{d(X)}\ \alpha_{X_k}(\varphi^\omega)(\ |\ |^2)$$

NOTE $\alpha_{X_k}(\varphi^\omega)(y) = \varphi^\omega(y)$

$\forall y \in \alpha_{X_k}(M)' \cap M^\omega$

$$(\because)\quad \alpha_{X_k}(\varphi^\omega)(y) = \alpha_{X_k}(\varphi)(\tau^\omega(y)) \underset{\substack{\alpha_{X_k}(M)'\cap M = Z(M)\\ \text{freeness}}}{\Uparrow}$$

$$\underset{\alpha_{X_k}\text{-inv.}}{=} \varphi(\tau^\omega(y)).$$

Hence

$$\sum_{k=1}^{m} \sum_{r\in J} \frac{d(X_k)}{d(X)} \underbrace{\| g_r \alpha_{X_k}(g_r) \|_{\varphi^\omega}^2}_{\text{very small.}} < m\varepsilon^2$$

$$\varphi^\omega(|g_r \alpha_{X_k}(g_r)|) = \varphi^\omega(\alpha_{X_k}(g_r)\,|\text{—''—}|)$$

$$\leq \varphi^\omega(\alpha_{X_k}(g_r))^{\frac{1}{2}} \varphi^\omega(|\quad|^2)^{\frac{1}{2}}$$

$$= \varphi^\omega(g_r)^{\frac{1}{2}} \|\quad\|_{\varphi^\omega}$$

$$J_0 := \{ r \in J \mid \sum_{k\geq 1} \frac{d(X_k)}{d(X)} \| g_r \alpha_{X_k}(g_r) \|_{\varphi^\omega}^2 < m\varepsilon \|g_r\|_{\varphi^\omega}^2$$

If. $r \in J_0$, then.

$$\left(\sum_{k\geq 1} \sqrt{\frac{d(X_k)}{d(X)}}\, |g_r \alpha_{X_k}(g_r)|_{\varphi^\omega} \right)^2$$

$$\sqrt{\frac{d(X)}{d(X_k)}}\sqrt{\frac{d(X_k)}{d(X)}}$$

$$\leq \cancel{m} \sum_{k\geq 1} \frac{d(X_k)}{d(X)} |g_r \alpha_{X_k}(g_r)|_{\varphi^\omega}^2$$

$$\leq \cancel{m} \sum_{k\geq 1} \frac{d(X_k)}{d(X)} \varphi(g_r) \|\quad\|_{\varphi^\omega}^2$$

$$< \cancel{m}^2\, \varphi(g_r)^2 \varepsilon$$

$$\to \cancel{\sum_{k\geq 1}}\; m \sum_{k\geq 1} \frac{d(X)}{d(X_k)} \varepsilon\, \varphi(g_r)^2$$

$$\doteqdot m^2 d(X)\, \varepsilon\, \varphi(g_r)^2$$

$$\leadsto \sum_{k\geq 1} |g_r \alpha_{X_k}(g_r)|_{\varphi^\omega} < m\, d(X)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \varphi(g_r).$$

$$\sum_{r\in J\setminus J_0} |g_r|_{\varphi^\omega} \leq \frac{1}{m\varepsilon} \sum_r \sum_{k\geq 1} \frac{d(X_k)}{d(X)} \|\quad\|^2 < \frac{1}{m\varepsilon} m\varepsilon^2 = \varepsilon$$

$$g_0 := \sum_{r\in J\setminus J_0} g_r.$$

Lem 2.1 (Support estim).

$e \in M_\omega$ proj.

$x \in \mathcal{C}$.

$s(\phi_x^\alpha(e)) :=$ the supp. proj of $\phi_x^\alpha(e)$ $\left(\begin{smallmatrix}\text{NOTE}\\ \in M_\omega\end{smallmatrix}\right)$.

Then $\forall\, \varphi \in M_*$ state

$$\varphi^\omega(s(\phi_x^\alpha(e))) \leq 2d(x)^2 \alpha(\varphi)^\omega(e).$$

Proof.

Set $q_x := c_{\bar{x},x} R_x^\alpha R_x^{\alpha *} c_{\bar{x},x}^*$ (proj in M).

$u := 2q_x - 1$ $\in$ (self-adj unitary in M)

$p := \alpha_{\bar{x}}(e) \vee u\alpha_{\bar{x}}(e)u^*$ $\in$ (M^ω proj. in M^ω)

$\rightsquigarrow\ upu^* = p$

$\rightsquigarrow\ pq_x = q_x p$.

Then

$$\underbrace{p^\perp q_x}_{\nearrow \text{ proj}} \overset{\uparrow M_\omega}{\phi_x^\alpha(e)} = p^\perp c_{\bar{x},x} R_x^\alpha \phi_x^\alpha(e) R_x^* c_{\bar{x},x}^*$$

$$= p^\perp c_{\bar{x},x} R_x^\alpha R_x^{\alpha *} c_{\bar{x},x}^* \alpha_{\bar{x}}(e)\, c_{\bar{x},x} R_x^\alpha R_x^{\alpha *} c_{\bar{x},x}$$

$$= p^\perp q_x \alpha_{\bar{x}}(e) q_x$$

$$= q\, p^\perp \alpha_{\bar{x}}(e) q$$

$$= 0. \qquad p \geq \alpha_x(e)$$

$\rightsquigarrow\ s(\phi_x^\alpha(e)) \leq 1 - p^\perp q$

$$q\, s(\phi_x^\alpha(e))\, q \leq q(1 - p^\perp q)q$$

$$\begin{array}{ccc} \| & & \| \\ s(\phi_x^\alpha(e))\, q & & (1-p^\perp)q \\ & & \| \\ & & pq \leq p \end{array} \quad \text{— ①}$$

Let $\psi := \alpha_{\bar{x}}(\alpha_x(\varphi))$

$\rightarrow \psi^\omega$ commuting $\alpha_{\bar{x}}(M_\omega)$ & u

(∵)

$$q\, \alpha_{\bar{x}}(\alpha_x(\varphi)) = c_{\bar{x},x} R_x^\alpha R_x^{\alpha *} \overset{\bar{x}\cdot x \to \bar{x}\cdot x}{\alpha_{\bar{x}\cdot x}(\varphi)}\, c_{\bar{x},x}^*$$

$$= c_{\bar{x},x} \alpha_{\bar{x}\cdot x}(\varphi) R_x^\alpha R_y^{\alpha *} - .$$

$$= \alpha_{\bar{x}}(\alpha_x(\varphi))\, q$$

On the one hand,

$$\psi^\omega(p) = \alpha_{\bar{x}}(\alpha_x(\varphi^\omega))(\overset{\alpha_{\bar{x}}(e) \vee u\alpha_{\bar{x}}(e)u^*}{\overset{\downarrow}{p}}).$$

tracial!

$$\leqq \alpha_{\bar{x}}(\alpha_x(\varphi^\omega))(\alpha_{\bar{x}}(e)) + \alpha_{\bar{x}}(\alpha_x(\varphi^\omega))(\cancel{u}\,\alpha_{\bar{x}}(e)\,\cancel{u^*})$$

$$= \alpha_x(\varphi^\omega)(e) + \alpha_x(\varphi^\omega)(e)$$

$$= 2\alpha_x(\varphi^\omega)(e). \quad \text{—②}$$

On the other hand,

$$\psi^\omega(S(f_x^\alpha(e))g) = \alpha_{\bar{x}}(\alpha_x(\varphi^\omega))(S(\ )\, c_{\bar{x}x}^* R_x R_x^* c_{\bar{x},x})$$

$$= \frac{1}{d(x)^2}\varphi^\omega(R_x^* c_{\bar{x},x}^* \underset{M_\omega}{\underset{\cap}{S(\ )}} c_{\bar{x},x} R_x)$$

$$= \frac{1}{d(x)^2}\varphi^\omega(S(\ ))$$

Hence

$$\varphi^\omega(S(\ )) = d(x)^2\psi^\omega(S(\ )g) \overset{①}{\leqq} d(x)^2\psi^\omega(p) \leqq 2d(x)^2\alpha_x(\varphi^\omega)(e) \blacksquare$$

Lem. 2.12

$(\alpha, c) : \mathcal{C} \curvearrowright M$ cent. free.

$\varphi \in M_*$: faitful state α-inv. on $Z(M)$.

$\forall F \subset\subset \mathrm{Irr}\,\mathcal{C} \setminus \{1\}$, $\forall Q \subset M^\omega$ countable vN.

$\forall \delta > 0$.

$\exists \{e_r\}_{r=0}^n$: part. of 1 in $Q' \cap M_\omega$

s.t.

- $|e_0|_{\varphi^\omega} < \delta$
- $e_r \alpha_x(e_r) = 0 \quad \forall r \geqq 1 \quad \forall x \in F.$

Proof. (Ocneanu).

WMA $\begin{cases} \alpha_x(Q) \subset Q & \forall x \in \mathrm{Irr}\,\mathcal{C} \\ M \subset Q \end{cases} \rightsquigarrow \alpha_x(Q) \subset Q \quad x \in \mathcal{C}.$

Claim I

$\forall \mu > 0 \quad 0 \neq \forall f \in Q' \cap M_\omega$ proj

$\exists f' \in Q' \cap M_\omega \quad 0 \neq f' \leqq f$ &

$$\sum_{x \in F} |f'\alpha_x(f')|_{\varphi^\omega} < 2\mu |f'|_{\varphi^\omega}$$

(ii) Claim 1.

$\tilde{Q} := Q \vee \{\alpha_x(f) \mid x \in \mathrm{Irr}\}''$.

By Lem 2.10 we have $q_0, q_1, \dots, q_n \in \tilde{Q}' \cap M^\omega$.

- $|q_0|_{\varphi^\omega} < \frac{1}{2} |f|_{\varphi^\omega}$
- $\sum_{x \in F} |q_r \alpha_x(q_r)|_{\varphi^\omega} < \mu |f|_{\varphi^\omega} |q_r|_{\varphi^\omega}$ $\quad {}^\forall r = 1 \dots n$

Put $\bar{q}_r := q_r f \ (\leq f)$ $\quad r = 0 \dots n$.

$ \overset{\shortparallel}{f q_r} \in \tilde{Q}^{\omega\prime} \cap M^\omega$

$\bar{q}_r \alpha_x(\bar{q}_r) = f q_r \alpha_x(q_r) \alpha_x(f)$

$\rightarrow |\bar{q}_r \alpha_x(\bar{q}_r)|_{\varphi^\omega} \leq |q_r \alpha_x(q_r)|_{\varphi^\omega}$

Suppose. ${}^\forall r = 1 \dots n$

$$\sum_{x \in F} |\bar{q}_r \alpha_x(\bar{q}_r)|_{\varphi^\omega} \geq 2\mu |\bar{q}_r|_{\varphi^\omega}$$

Then $\rightsquigarrow$ $\sum_{r=1}^{n} \sum_{x \in F} |\bar{q}_r \alpha_x(\bar{q}_r)| \geq 2\mu \sum_{r=1}^{n} |\bar{q}_r|_{\varphi^\omega}$

$$\begin{array}{ccc} \sum_{r=1}^{n} \sum_{x \in F} |\bar{q}_r \alpha_x(\bar{q}_r)| & \geq & 2\mu \sum_{r=1}^{n} |\bar{q}_r|_{\varphi^\omega} \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$=$} \\ \sum_{r \geq 1}^{n} \sum_{x \in F} |q_r \alpha_x(q_r)|_{\varphi^\omega} & & 2\mu \varphi^\omega((1-q_0) f) \\ \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} \\ \sum_{r=1}^{n} \mu |f|_{\varphi^\omega} |q_r|_{\varphi^\omega} \leq \mu |f|_{\varphi^\omega} & & 2\mu(\varphi^\omega(f) - \varphi^\omega(q_0)) > \frac{\mu}{\cancel{2}} |f|_{\varphi^\omega} \end{array}$$

$\exists r_0 \geq 1 \ \sum_{x \in F} |\bar{q}_{r_0} \alpha_x(\bar{q}_{r_0})| < 2\mu |\bar{q}_{r_0}|_{\varphi^\omega}$

$\rightarrow \bar{q}_{r_0} \neq 0$ & $\bar{q}_{r_0} \leq f$ $\quad \boxtimes$

Claim 2.

${}^\forall \mu > 0 \ {}^\forall \underset{\neq 0}{f} \in Q' \cap M_\omega$ proj ~~≠0~~.

$\exists \underset{\neq 0}{e} \in Q' \cap M_\omega$ ~~n~~proj

s.t.

(1) $e \leq f$

(2) $\sum_x |e \alpha_x(e)|_{\varphi^\omega} \leq \mu |e|_{\varphi^\omega}$ $({}^\forall x \in F)$

(3) $|e|_{\varphi^\omega} \geq (1 + 4|F|_\sigma)^{-1} |f|_{\varphi^\omega}$

Proof of Claim 2.

Take a maximal proj $\underset{\neq 0}{e}$ with (1), (2). $\quad \overset{\text{by}}{\Leftarrow}$ Clam 1.

$\rightsquigarrow e \vee \bigvee_{x \in F \cup \bar{F}} s(\phi_x^\alpha(e)) \vee (1-f) = 1$

(ii) If $\underset{\underset{\tilde{Q}' \cap M_\omega}{\cap}}{p} \perp \mathrm{LHS} \overset{\leq f}{\neq 0}$

$\sum_x |p \alpha_x(p)|_{\varphi^\omega} \leq \mu |p|_{\varphi^\omega}$

$f \geq \bar{e} := e + p \geq e.$

~~$(e+p)\ \alpha_x$~~(e)

$p\, \phi_x^{\alpha}(e)\ p = 0$

$\quad \|$

$\phi_x^{\alpha}(\alpha_x(p)\, e\, \alpha_x(p))$

$\to e\,\alpha_x(p) = 0 \quad x \in F \cup \bar{F}.$

&

$\phi_x^{\alpha}(\alpha_x(e)\, p\, \alpha_x(e))$

$= e\ \phi_x^{\alpha}(p)\ e$

$= \underset{\substack{\uparrow \\ M_\omega}}{e}\ \underset{\substack{\uparrow \\ M}}{\underbrace{R_x^{*}\, e_{\bar{x},x}^{x}}}\ \alpha_{\bar{x}}(p) \ \cdots$

$= 0.$

$\to p\,\alpha_x(e) = 0$

Hence

$\bar{e}\ \alpha_x(\bar{e}) = e\,\alpha_x(e) \underset{\substack{\uparrow \\ (M^{\omega})_{\varphi^\omega}}}{+} p\,\alpha_x(p) \qquad \alpha_x M_\omega$

freeness
φ inv.

$|\bar{e}\ \alpha_x(\bar{e})|_{\varphi^\omega} \leq |e\alpha_x(e)|_{\varphi^\omega} + |p\alpha_x(p)|_{\varphi^\omega}$

$\sum \quad \leq \mu|e|_{\varphi^\omega} + \mu|p|_{\varphi^\omega} = \mu|\bar{e}|_{\varphi^\omega}.$

1 By ~~Lem.~~

$\varphi^\omega|_{M_\omega}$ trace

$1 \overset{\forall}{\leq} \varphi^\omega(e) + \sum_{x \in F \cup \bar{F}} \varphi^\omega(s(\phi_x^{\alpha}(e))) + \varphi^\omega(1-f)$

$\leq \varphi^\omega(e) + \sum_{x \in F \cup \bar{F}} 2\, d(x)^2 (\varphi^\omega)(e) + -$

([Lem 2.11]) $\Longrightarrow \alpha$-inv [illegible]

$= (1 + 4|F|_0)\, \varphi^\omega(e). + 1 - \varphi^\omega(f)$

Fix $\delta > 0$ & $m \in \mathbb{N}\,(1 - (1+4|F|_0)^{-1})^m < \delta.$

Claim 3.

$^\forall\delta > 0.\ \ ^\exists e_0\, e_1 \cdots e_m \in Q' \cap M_\omega$ part. of 1

$^\forall\mu > 0.$

(1) $|e_0|_{\varphi^\omega} < \delta$

(2) $\sum_{x \in F} |e_k\, \alpha_x(e_k)|_{\varphi^\omega} < \mu|e_k|_{\varphi^\omega}$

(∵) Claim 3.

Let $f_1 = 1$. By induction $^\exists e_k \leq f_{k}$

- $e_k \leq f_k \qquad f_{k+1} := f_k - e_k.$
- $\sum_x |e_k\alpha_x(e_k)| \leq \mu|e_k|_{\varphi^\omega}$
- $|e_k|_{\varphi^\omega} \geq (1+4|F|_0)^{-1}|f_k|_{\varphi^\omega}$

$|f_{k+1}|_{\varphi^\omega} \leq (1-(1+4|F(\sigma))^{-1}|f_k|_{\varphi^\omega}$

$\to |f_{m+1}|_{\varphi^\omega} \leq (\qquad)^{-m} \underset{\overset{\shortparallel}{1}}{|f_1|_{\varphi^\omega}} < \delta.$

Put $e_0 := f_{m+1}$. ☒.

Cla~~ims~~4.

Thus by Claim 3. $\forall n = 1.2\ldots$ $\quad \exists\{e_k^n\}_{k=1}^{\textcircled{n}} \in Q \cap M_\omega$ part of 1

$\boxed{\text{const to } \underline{n}} \xrightarrow{} \textcircled{n}$

$\cdot\ |e_0^n|_{\varphi^\omega} \leq \delta$

$\cdot \sum_{x\in F} |e_k^n \alpha_x(e_k^n)|_{\varphi^\omega} < \frac{1}{n} |e_k^n|_{\varphi^\omega}$

$e_k^1 = (\circ, \circ, \circ, \circ, \circ \cdots\cdots)^\omega$

$e_k^2 = (\cdot, \cdot, \cdot, \cdot, \cdots\cdots)^\omega$

$\vdots$

make a new sequence

a~~rrange~~ ~~&~~

by taking elements "diagonally". ☒

No.

Lem. 2.13

- $(\alpha . c) : \mathcal{C} \curvearrowright N$ ← not necessarily sep N_*

- $K \subset\subset \mathrm{Irr}\,\mathcal{C}$

Suppose a proj $e \in N$ satisfies

- $e\,\alpha_x(e) = 0 \quad \forall x \in (\bar{K} \cdot K) \setminus \{1\}$

  $e \in \square_\alpha' \cap N$
- $e ~~\#~~ T^\alpha = T^\alpha e \quad \forall T$ morphism in $\mathcal{C}$

  $e\, C_{x,y} = C_{x,y}\, e$

Then the following holds:

(1) $\alpha_x(e)\,\alpha_y(e) = 0 \quad \forall x \neq y \in K$

(2) $\{d(x)^2 \phi_{\bar{x}}^\alpha(e)\}_{x \in K}$ mutually ortho. projections

(3) $d(x)^2 \alpha_x(e)\, C_{x,\bar{x}} \bar{R}_x^\alpha \bar{R}_x^{\alpha *} C_{x,\bar{x}}\, \alpha_x(e)$

$= \alpha_x(e)$

(1) $\alpha_{\bar{x} \otimes y}(e) = \sum_{z,S} S^\alpha \alpha_z(e) S^{\alpha *}$

$S \in \mathrm{ONB}(z, \bar{x} \otimes y)$.

$z \prec \bar{x} \otimes y$.

$z \neq 1$.

since $x \neq y$

$\rightsquigarrow e\,\alpha_{\bar{x} \otimes y}(e) = \sum S^\alpha e\, \alpha_z(e) S^{\alpha *} = 0$.

by asump.

$\rightsquigarrow e\,\alpha_{\bar{x}}(\alpha_y(e)) = 0$.

$\phi_x^\alpha(\alpha_x(e)\,\alpha_y(e)\,\alpha_x(e))$

$= e\, \phi_x^\alpha(\alpha_y(e))\, e$

$= e\, R_x^{\alpha *} C_{\bar{x},x}^* \alpha_{\bar{x}}(\alpha_y(e))\, C_{\bar{x},x} R_x^\alpha e$

$= 0$.

(2) $\alpha_{\bar{x}}(\alpha_x(e)) = C_{\bar{x},x}\, \alpha_{\bar{x} \otimes x}(e)\, C_{\bar{x},x}^*$

$= C_{\bar{x},x} \sum_{z,S} S^\alpha \alpha_z(e) S^{\alpha *} C_{\bar{x},x}^*$

$S : z \to \bar{x} \otimes x$

$e\,\alpha_z(e) = 0$ if $z \neq 1$

$e\,\alpha_{\bar{x}}(\alpha_x(e)) = C_{\bar{x},x} R_x^\alpha\, e\, R_x^{\alpha *} C_{\bar{x},x}^*$

$= \alpha_{\bar{x}}(\alpha_x(e))\, C_{\bar{x},x} R_x^\alpha R_x^{\alpha *} C_{\bar{x},x}^*$

$\xrightarrow{\phi_{\bar{x}}^\alpha}$ $\phi_{\bar{x}}^\alpha(e)\,\alpha_x(e) = \alpha_x(e)\, \underset{\frac{1}{d(x)^2}}{\underset{\shortparallel}{\phi_{\bar{x}}^\alpha(C_{\bar{x},x} R_x^\alpha R_x^{\alpha *} C_{\bar{x},x}^*)}}$

$d(x)^2 \phi_{\bar{x}}^\alpha(e)\,\alpha_x(e) = \alpha_x(e)$.

Easy to check $\phi_X^\alpha(e)$ commutes with T^α & $C_{X,Y}$,

Hence

$$d(X)^4\,\phi_{\bar X}^\alpha(e)^2 = \phi_{\bar X}^\alpha(e)\,\bar R_X^{\alpha *} c_{X,\bar X}^{*}\,\alpha_X(e)\,c_{X,\bar X}\,\bar R_X^\alpha$$

$$= \bar R_X^{\alpha *} c_{X,\bar X}^{*}\,\frac{\alpha_X(e)}{d(X)^2}\,c_{X,\bar X}\,\bar R_X^\alpha$$

$$= \frac{\phi_{\bar X}^\alpha(e)}{d(X)^2}$$

$\rightarrow$ $d(X)^2\,\phi_{\bar X}^\alpha(e)$ proj.

Orthogonality $X \neq Y \in K$

$$\phi_{\bar X}^\alpha(e)\,\phi_{\bar Y}^\alpha(e) = \phi_{\bar X}^\alpha(\underbrace{e\,\alpha_{\bar X}(\phi_{\bar Y}^\alpha(e))}_{\substack{\parallel \\ 0}}).$$

Inded,

$$e\,\alpha_{\bar X}(\phi_{\bar Y}^\alpha(e)) = e\,\alpha_{\bar X}(\bar R_Y^{\alpha *} c_{Y,\bar Y}^{*}\,\alpha_Y(e)\,c_{Y,\bar Y}\,\bar R_Y^\alpha)$$

$$= \alpha_{\bar X}(\bar R_Y^{\alpha *} c_{Y,\bar Y}^{*})\,\underbrace{e\,\alpha_{\bar X}(\alpha_Y(e))}_{\substack{\parallel \\ 0}} \cdots$$

(3) ~~We know~~ Since.

$$d(X)^2\phi_{\bar X}^\alpha(e) = d(X)^2\,\bar R_X^{\alpha *} c_{X,\bar X}^{*}\,\alpha_X(e)\,c_{X,\bar X}\,\bar R_X^\alpha$$

$\rightarrow$ $d(X)\,\alpha_X(e)\,c_{X,\bar X}\,\bar R_X^\alpha$ p.i.

$\rightarrow$ $d(X)^2\,\alpha_X(e)\,c_{X,\bar X}\,\bar R_X^\alpha\,\underset{\substack{\uparrow \\ \text{proj}}}{\bar R_X^{\alpha *}}\,c_{X,\bar X}^{*}\,\alpha_X(e) \leqq \alpha_X(e)$

= holds indeed

$\xrightarrow{\phi_X^\alpha}$ $d(X)^2\,e\,\phi_X^\alpha(c_{X,\bar X}\,\bar R_X^\alpha\,\bar R_X^{\alpha *}\,c_{X,\bar X}^{*})\,e \leqq e$

$\parallel$

e

∎

Section 3. Rohlin tower construction.

Rohlin: $\mathbb{Z} \overset{\alpha}{\curvearrowright} L^\infty(X,\mu)$ ergodic.

$\underset{\alpha}{e_1 \curvearrowright e_2} \underset{\alpha}{\curvearrowright} \cdots , e_m$ part of 1

Connes: $\mathbb{Z} \curvearrowright M_\omega$ cent. free.

Ocneanu: $\underset{\text{amenable}}{\Gamma} \curvearrowright M_\omega$ cent free

(F,δ)-inv. K

i.e. $|F\cdot K \triangle K| < \delta |K|$

$^\exists \{e_g\}_{g\in K}$ part of 1 in M_ω

$e_g = 0$ $(g \notin K)$.

$$\sum_{h\in F}\sum_{g\in K} |\alpha_h(e_g) - e_{hg}|_{\varphi^\omega} \le \underset{\substack{\uparrow\\ \text{const.}}}{C}\,\delta^{\frac{1}{2}}$$

Thm 3.1 (Rohlin tower).

- $(\alpha, c) : \underset{\substack{\uparrow\\ \text{amenable}}}{\mathcal{C}} \curvearrowright M$ cent. free. (M: vN alg.)
- $\varphi \in M_* :$ faithful state $\varphi|_{Z(M)}$ α-inv.
- $^\forall F \subset\subset \operatorname{Irr}\mathcal{C}$, $^\forall \delta > 0$.
  $F = \bar{F}$
  $^\forall K \subset\subset \operatorname{Irr}\mathcal{C}$ (F,δ)-inv.
- $^\forall Q \subset M^\omega$ countable vN alg.

Then $^\exists E_X \in M^\omega$ projection with $X \in \mathcal{C}$

s.t.

(1) (Naturality)

$$E_Y T^\alpha = T^\alpha E_X \quad {}^\forall X, Y \in \mathcal{C},\ {}^\forall T \in \mathcal{C}(X,Y)$$

(2) (Supported on K)

$$E_X = 0 \quad {}^\forall X \in \operatorname{Irr}\mathcal{C} \setminus K.$$

(3)

$$E_X \in \alpha_X(Q)' \cap M^\omega \quad {}^\forall X \in \operatorname{Irr}\mathcal{C}$$

(4) (Splitting property)

$$\tau^\omega(E_x x) = \tau^\omega(E_x)\tau^\omega(x)$$

$$\forall x \in \mathcal{C},\ \forall x \in Q.$$

(5) ("~~Partition of 1~~")

$$\{ d(x)^2 \bar{R}_x^{\alpha *} C_{x,\bar{x}}^* E_x C_{x,\bar{x}} \bar{R}_x^\alpha \}_{x \in K}$$

orthogonal projections.

(6) (approx. part. of 1).

$$\sum_{x \in K} d(x)^2 \varphi^\omega(\bar{R}_x^{\alpha *} C_{x,\bar{x}}^* E_x C_{x,\bar{x}} \bar{R}_x^\alpha) \geqq 1-\delta^{\frac{1}{2}}$$

(7) (approx. equiv).

$$\sum_{x \in F} \sum_{y \in \mathrm{Irr}\,\mathcal{C}} d(x)^2 d(y)^2 \left| \alpha_x(E_y) - C_{x,y} E_{x \circ y} C_{x,y}^* \right|_{\alpha_x(\alpha_y(\varphi^\omega))}$$

$$\leqq 6\delta^{\frac{1}{2}} |F|_\sigma$$

(8) (Resonance prop)

$$E_x C_{x,\bar{x}} \bar{R}_x^\alpha E_y = \frac{\delta_{x,y}}{d(x)} d_x(C_{\bar{x},x} R_x^\alpha) E_x$$

$$\forall x, y \in \mathrm{Irr}\,\mathcal{C}.$$

We will call such $(E_x)_x$ a <u>Rohlin tower</u> <u>along with</u> K. $\overset{\shortparallel}{E}$

$$J := \{ (E_x)_x \mid \overset{\text{satisfies}}{(1),(2),(3),(4),(5),(8)} \}$$

<u>Lem.</u> $0 = (0)_x \in J$. $J \neq \phi$.

$$a_E := \frac{1}{|F|_\sigma} \sum_x \sum_{y \in \mathrm{Irr}\,\mathcal{C}} d(x)^2 d(y)^2 \left| \alpha_x(E_y) - C_{x,y} E_{x \circ y} C_{x,y}^* \right|_{\alpha_x(\alpha_y(\varphi^\omega))}$$

$$b_E := \sum_x d(x)^2 \varphi^\omega(\bar{R}_x^{\alpha *} \cdots \bar{R}_x^\alpha) \leqq 1.$$

$$= \sum_x d(x)^2 \varphi^\omega(E_x).$$

<u>Lem.</u>

Let $E = (E_x)_x \in J$ & $b_E < 1-\delta^{\frac{1}{2}}$

$\exists E' \in J$ s.t.

- $a_{E'} - a_E \leqq 6\delta^{\frac{1}{2}}(b_{E'} - b_E)$
- $0 < \frac{\delta^{\frac{1}{2}}}{2} \sum_{x \in K} d(x)^2 \varphi^\omega(|E_x' - E_x|) \leqq b_{E'} - b_E$

proof (Sketch) (due to Ocneanu)

. $|e_0| < \delta_1$ very small $\delta_1 < \delta$.

$e_r \alpha_x(e_r) = 0$ $r \geq 1$. $x \in \overline{K}_1 \cdot K_1 \setminus \{1\}$

$K_1 = K \cup (F \cdot K)$

$\rightarrow$

$$f_r := \sum_{x \in F} d(x)^2 \underset{\text{proj}}{\phi_{\bar{x}}^{\alpha}(e_r)} \in M_\omega \cap Q' \quad r \geq 1.$$

$$E_x'^{\,r} := \underset{\substack{\uparrow \\ \text{destroying part}}}{\overset{\text{old one}}{E_x} f_r^{\perp}} + \underset{\substack{\uparrow \\ \text{building part}}}{\alpha_x(e_r)} \quad x \in K.$$

We can show $\exists r \geq 1$. the desired ineqs holds. ∎

Proof of Thm.

$$E_1 \leqq E_2 \text{ in } J \overset{\text{dfn}}{\Longleftrightarrow} E_1 = E_2 \text{ or } \left(\begin{array}{l} a_{E_2} - a_{E_1} \leq \delta^{\frac{1}{2}} (b_{E_2} - b_{E_1}) \\ b < \cdots \end{array} \right.$$

$\leqq$ is an order.

inductive ?

$(E^\lambda)_{\lambda \in \Lambda}$ totally ordered.

b_{E^λ} ↗ increasing in $\mathbb{R}_+$ $[0, 1]$.

$\boxed{b_{E^\lambda}}$ ↗ $b \in [0, 1]$.

↑ Cauchy

$$\frac{\delta^{\frac{1}{2}}}{2} \sum_{x \in K} d(x)^2 \varphi^\omega(|E_x^\lambda - E_x^\mu|) \xrightarrow[\lambda, \mu]{} 0$$

$\rightarrow$ E_x^λ $\overset{s^*}{\text{Cauchy}}$.

$$E_x^\lambda \xrightarrow{s^*} E_x \in J.$$

Let $E = (E_x)_x \in J$ maximal

By Lem $b_E \geqq 1 - \delta^{\frac{1}{2}}$

∎

Section 4. Classification

Given

- $(\alpha, c^\alpha) : \mathcal{C} \curvearrowright M$ cent free
- $(\beta, c^\beta) : \mathcal{C} \curvearrowright M$ cent free
- They are approx. unitary equiv.

  i.e. $\forall x \in \mathrm{Irr}\,\mathcal{C}$. $\exists u_x \in M^\omega$

  $$u_x \alpha_x(\varphi^\omega) u_x^* = \beta_x(\varphi^\omega). \quad \forall \varphi$$

  $$\Leftrightarrow \lim_{n \to \omega} \| u_x^{(n)} \alpha_x(\psi) u_x^{(n)*} - \beta_x(\psi) \| = 0.$$

  $$\to u_x \alpha_x(a) u_x^* = \beta_x(a) \quad \forall x \in M.$$

- (α, c^α) has $\varphi \in M_*^+$ faith st.

  $$\varphi|_{Z(M)}$$

  $\alpha|_{Z(M)}$ pmp action

  $\|$

  $\beta|_{Z(M)}$.

Thm. 4.1

$(\alpha, c^\alpha) \underset{\text{strong}}{\sim} (\beta, c^\beta)$

※ i.e.

strongness

$\downarrow$

- $\exists\, \theta \in \mathrm{Aut}(M)$ $\boxed{\text{approx inner}}$
- $\exists\, v_x \in M$ unitaries $(x \in \mathcal{C})$.

s.t.

$$\begin{cases} \cdot\ \theta \cdot \alpha_x \cdot \theta^{-1} = \mathrm{Ad}\, v_x \cdot \beta_x & \forall x \in \mathcal{C} \\ \cdot\ \theta(c^\alpha_{x,y}) = v_x \beta_x(v_y) c^\beta_{x,y} v_{x \cdot y}^* & \forall x, y \in \mathcal{C} \\ \cdot\ \theta(T^\alpha_x) = v_y T^\beta v_x^*. & \forall x, y \in \mathcal{C}. \\ & \forall T \in \mathcal{C}(X, Y) \end{cases}$$

Ra
Lem. 4.2.

$\exists\ u_x \in M^\omega$ $(x \in \mathcal{C})$ unitary

s.t.

· $\mathrm{Ad}\, u_x \cdot \alpha_x(\psi^\omega) = \beta_x(\psi^\omega)$ $\quad \forall x \in \mathcal{C}\ \forall \psi \in M_*$

· $u_Y T^\alpha u_X^* = T^\beta$ $\quad T : X \to Y$

(∵)

Take for $x \in \mathrm{Irr}\,\mathcal{C}$. s.t.

$\mathrm{Ad}\, u_x \cdot \alpha_x(\psi^\omega) = \beta_x(\psi^\omega)$ $\quad \forall \psi$

Then $\forall X \in \mathcal{C}$. set

$$u_X := \sum_{z \in \mathrm{Irr}\,\mathcal{C}} S^\beta u_z S^{\alpha *}$$

$S \in \mathrm{ONB}(z, X)$. $\boxed{}$

Then we set $\gamma_x := \mathrm{Ad}\, u_x \cdot \alpha_x$

$c^\gamma_{x,y} := u_x \alpha_x(u_y) c^\alpha_{x,y} u_{x\cdot y}^*$

$T^\gamma := u_Y T^\alpha u_X^* = T^\beta$

$(\gamma, c^\gamma) : \mathcal{C} \curvearrowright M^\omega$ cocycle action.

$$\begin{cases} \gamma_x(\varphi^\omega) = \beta_x(\varphi^\omega) & \forall \varphi \in M_* \\ \gamma_x(x) = \beta_x(x) & \forall x \in M \\ T^\gamma = T^\beta & \end{cases}$$

~~Want~~ But $c^\gamma_{x,y} \neq c^\beta_{x,y}$. ingeneral ...

We want to perturb $(\gamma, c^\gamma) \xrightarrow[\text{close to}]{} (\beta, c^\beta)$.

Rohlin tower works well!

Let (F, δ) -inv. $K \subset\subset \mathrm{Irr}\,\mathcal{C}$.

large $\quad$ small > 0.

$Q \subset M^\omega$ enough large ($u_x \in Q$ etc).

$\rightsquigarrow$ $E = (E^\alpha_x)_x$ Rohlin tower along with K.

$$\alpha_x(E^\alpha_y) \overset{6\delta^{\frac{1}{2}}}{\sim} c^\alpha_{x,y} E_{x\cdot y} c^{\alpha *}_{x,y}$$

$x \in F$. $y \in \mathrm{Irr}\,\mathcal{C}$

~~To simp~~lify.

Set $E_x^\gamma := v_x E_x^\alpha v_x^*$. $x \in \mathcal{C}$.

→ ~~Take Θ enou.~~

Lem. 4.3

$E^\gamma = (E_x^\gamma)_x$ satisfies Thm 3.1 conditions.

(∵) we only check the resonance

$$E_x^\gamma C_{x.\bar{x}}^\gamma \bar{R}_x^\gamma E_y^\gamma$$

$$= u_x E_x^\alpha \cancel{u_x^*} \cdot \cancel{u_x} \alpha_{\cancel{x}}(u_{\bar{x}}) C_{x.\bar{x}}^\alpha u_{x\circ\bar{x}}^*$$

$u_{x\circ\bar{x}} \bar{R}_x^\alpha u_1^*$

$u_y E_y^\alpha u_y^*$

$$= u_x \alpha_x(u_{\bar{x}}) E_x^\alpha C_{x.\bar{x}}^\alpha \bar{R}_x^\alpha \quad u_y E_y^\alpha u_y^*$$

$$= u_x \alpha_x(u_{\bar{x}} \alpha_{\bar{x}}(u_y)) E_x^\alpha C_{x.\bar{x}}^\alpha \bar{R}_x^\alpha E_y^\alpha u_y^*$$

$$= \frac{\delta_{x.y}}{d(x)} u_x \alpha_x(u_{\bar{x}} \alpha_{\bar{x}}(u_x)) \cdot \alpha_x(C_{\bar{x}.x}^\alpha R_x^\alpha) E_x^\alpha u_x^*$$

$$= \frac{\delta_{x.y}}{d(x)} u_x \alpha_x(C_{\bar{x}.x}^\gamma v_{\bar{x}.x} R_x^\alpha) E_x^\alpha u_x^*$$

$$\gamma_x(C_{\bar{x}.x}^\gamma R_x^\gamma) E_x^\gamma$$

~~Now~~ We assume $b_E = 1$ for simplicity.

$$v_x := \sum_{y \in \mathrm{Irr}\,\mathcal{C}} d(y)^2 \bar{R}_y^{\gamma *} C_{y.\bar{y}}^{\gamma *} \gamma_y(C_{\bar{y}.x}^\beta C_{\bar{y}.x}^{\gamma *}) E_y^\gamma C_{y.\bar{y}}^\gamma \bar{R}_y^\gamma.$$

$\boxed{I = 4}$!

たthe foll. $C_{x.y} := C_{x.y}^\gamma$

$d_{x.y} := C_{x.y}^\beta$

Lem. 4.4.

- v_x unitary
- $v_x \gamma_x(\psi^\omega) = \gamma_x(\psi^\omega) v_x$
- $v_x T^\gamma = T^\beta v_x$

(∵)

$$v_x \gamma_x(\psi^\omega)$$

$$= \sum_y d(y)^2 \bar{R}_y^{\gamma *} C_{y.\bar{y}}^{\gamma *} \gamma_y(C_{\bar{y}.x}^\beta C_{\bar{y}.x}^{\gamma *}) E_y^\gamma C_{y.\bar{y}}^\gamma \bar{R}_y^\gamma \gamma_x(\psi^\omega)$$

$d(y)^2$

$\gamma_y \gamma_{\bar{y}} \gamma_x(\psi^\omega)$

commutes

$$= \sum d(y)^4 \bar{R}_y^{\gamma *} C_{y.\bar{y}}^* \gamma_y(C_{\bar{y}.x}^\beta C_{\bar{y}.x}^{\gamma *} \gamma_{\bar{y}} \gamma_x(\psi^\omega)) E_y^\gamma \text{---}$$

$$\cancel{\gamma_X}$$

$$\gamma_X(N_Y) = \sum_Z d(Z)^2 \gamma_X(\overline{R}_Z^{\gamma *} C_{Z,\overline{Z}}^{*} \gamma_Z(d_{\overline{Z},Y} C_{\overline{Z},Y}^{*}))$$

$$\cdot \boxed{\gamma_X(E_{\cancel{Z}}^{\gamma})}$$

$$\cdot \gamma_X(C_{\cancel{Z},\overline{Z}} \overline{R}_Z^{\gamma})$$

$$\overset{\delta^{\frac{1}{2}}}{\cancel{\approx}} \sum_Z d(Z)^2 \gamma_X(\overline{R}_Z^{\gamma *} C_{Z,\overline{Z}}^{*}) \cdot C_{X,Z} \gamma_{X \circ Z}(d C^{*}) \cancel{C_{X,Z}^{*}}$$

$$\cdot \cancel{C_{X,Z}} E_{X \circ Z} C_{X,Z}^{*}$$

$$\gamma_X(C_{Z,\overline{Z}} \overline{R}_Z^{\gamma})$$

$$= \sum_{Z,V} \sum_{S: V \to X \circ Z} d(Z)^2 \boxed{\gamma_X(\overline{R}_Z^{\gamma *} C_{Z,\overline{Z}}^{*}) C_{X,Z} S^{\gamma}}$$

$$\gamma_V(d_{\overline{Z},Y} C_{\overline{Z},Y}^{*}) E_V \quad \Big)^{*}$$

$$\boxed{S^{\cancel{\gamma} *} C_{X,Z}^{*} \gamma_X(\cancel{R}_{Z,\overline{Z}} \overline{R}_Z^{\gamma})}$$

ここで.

$$S : V \to X \circ Z.$$

$$\to \mathrm{Fr}(S) : X \to V \circ \overline{Z}$$

$$\overset{\shortparallel}{T}$$

$$\left(\cancel{\overline{Z}} \frac{d(\cancel{V})^{\frac{1}{2}}}{d(X)^{\frac{1}{2}} d(Z)^{\frac{1}{2}}} C_{V,\overline{Z}} T^{\gamma} \right)$$

$$\frac{d(V)^{\frac{1}{2}}}{d(X)^{\frac{1}{2}} d(Z)^{\frac{1}{2}}} T = (S^{*} \circ 1_{\overline{Z}})(1_{\cancel{Z}} \circ \overline{R}_Z)$$

$$\cancel{X}$$

$$\downarrow$$

$$\frac{d(V)^{\frac{1}{2}}}{d(X)^{\frac{1}{2}} d(Z)^{\frac{1}{2}}} C_{V,\overline{Z}} T^{\gamma} = C_{V,\overline{Z}} [(S^{*} \circ 1_{\overline{Z}})(1_X \circ \overline{R}_Z)]^{\gamma}$$

$$= S^{*} C_{X \cdot Z,\overline{Z}} [(1_X \circ \overline{R}_Z)]^{\gamma}$$

$$= S^{*} C_{X,Z}^{*} \gamma_X(C_{Z,\overline{Z}}) C_{X,Z \cdot \overline{Z}} [\ ]^{\gamma}$$

$$= S^{*} C_{X,Z}^{*} \gamma_X(C_{Z,\overline{Z}} \overline{R}_Z^{\gamma})$$

$\gamma_X(v_Y)$

$$\approx \sum_{Z,V} \sum_{T: X\to V\cdot\bar{Z}} \frac{d(\bar{Z})\,d(V)}{d(X)}\, T^{\gamma *} c_{V,\bar{Z}}^{*}\, \gamma_V(d_{\bar{Z},Y}\, c_{\bar{Z},Y}^{*}) \enclose{circle}{E_V}$$

$$c_{V,\bar{Z}}\, T^{\gamma}\,.$$

~~$v_X\,\gamma_X(v_Y)$~~

~~$\approx \sum_{Z,V,T.} \frac{d(Z)\,d(V)}{d(X)}\, T^{\gamma *}\, \cancel{E}\, \underline{\gamma_{V\cdot\bar{Z}}(v_X)\, c_{V,\bar{Z}}^{*}}$~~

~~$= \sum_{Z,V,T} \frac{d(Z)\,d(V)}{d(X)}\, T^{\gamma *} c_{V,\bar{Z}}$~~

~~$\gamma_V(\gamma_{\bar{Z}}(v_X)\, d_{\bar{Z},Y}\, c_{\bar{Z},Y}^{*})\, E_V$~~

$$v_X = \sum_{\upsilon} d(\upsilon)^2\, \bar{R}_{\upsilon}^{\gamma *}\, c_{\upsilon,\bar{\upsilon}}^{*}\, \gamma_{\upsilon}(d_{\bar{\upsilon},X}\, c_{\bar{\upsilon},X}^{*}) \enclose{circle}{c_{\upsilon,\bar{\upsilon}}\bar{R}_{\upsilon}^{\gamma}}$$

$$\enclose{circle}{E_{\upsilon}}$$

$v_X\,\gamma_X(v_Y)$

$$\approx \sum_{Z,V,\upsilon,T} \cancel{R}\, \frac{d(\upsilon)^2 d(Z)\,d(V)}{d(X)}$$

$$\bar{R}_{\upsilon}^{\gamma *}\, c_{\upsilon,\bar{\upsilon}}^{*}\, \gamma_{\upsilon}(d_{\bar{\upsilon},X}\, c_{\bar{\upsilon},X}^{*})\, \cancel{E_{\upsilon}}$$

$$E_{\upsilon}\cdot\ \gamma_{\upsilon}(\gamma_{\bar{\upsilon}}(T^{\gamma *} c_{V,\bar{Z}}^{*}\, \gamma_V(d_{\bar{Z},Y}\, c_{\bar{Z},Y}^{*})))$$

$$E_{\upsilon} \longrightarrow c_{\upsilon,\bar{\upsilon}}\, \bar{R}_{\upsilon}^{\gamma}\, E_V$$

$\delta_{V,\upsilon}$ comm.

$$= \sum_{Z,V,T} \frac{d(V)^2 d(Z)}{d(X)}$$

$$\bar{R}_{\upsilon}^{\gamma *}\, c_{\upsilon,\bar{\upsilon}}^{*}\, \cancel{\gamma_{\upsilon}} \enclose{circle}{(d_{\bar{\upsilon},X}\, c_{\bar{\upsilon},X}^{*})}$$

$$\gamma_{V}(\gamma_{\bar{V}}\enclose{circle}{(T^{\gamma} c_{V,\bar{Z}}^{*}\, \gamma_V(d_{\bar{Z},Y}\, c_{\bar{Z},Y}^{*}))})$$

$$\gamma_V(\enclose{circle}{c_{\bar{\upsilon},V}\, R_V^{\gamma}})\, E_V^{\gamma}$$

$$c_{V,\bar{Z}}\, T^{\gamma}$$

← P40 上.

$\rightarrow$

$\upsilon_X \gamma_X(\mathbb{B}_Y) C_{X,Y} \upsilon_{X \circ Y}^*$

$$\approx \sum_{\mho, Z, T} \frac{d(\mho)^2 d(Z)}{d(X)}$$

$$\overline{R}_\mho^{\gamma *} C_{\mho,\overline{\mho}}^* \gamma_\mho \left(d_{\overline{\mho},Y} C_{\overline{\mho},X}^* \gamma_{\overline{\mho}} (T^{\gamma *} C_{\mho,\overline{Z}}^* \gamma_\mho (d_{\overline{Z},Y} C_{\overline{Z},Y}^*)) \right)$$

$$\cdot \boxed{\gamma_\mho (C_{\overline{\mho},\mho} \overset{\gamma}{R_\mho}) E_\mho \underset{\overset{\wedge}{C_{\mho,\overline{Z}} T^\gamma}}{} C_{X,Y} \upsilon_{X \circ Y}^*} \quad —①$$

$T : X \rightarrow \mho \circ Z$.

$E_\mho C_{\mho,\overline{Z}} \overline{T}^\gamma C_{X,Y} \upsilon_{X \circ Y}^*$

$$= E_\mho \boxed{C_{\mho,\overline{Z}} T^\gamma C_{X,Y}} \cdot \sum_V d(V)^2 \overline{R}_V^{\gamma *} C_{V,\overline{V}}^* \cancel{\gamma_{\overline{V},X}} \cdot \gamma_V (C_{\overline{V},X \circ Y} d_{\overline{V},X \circ Y}^*) E_V C_{V,\overline{V}} \overline{R}_V^\gamma$$

$$= E_\mho \sum_V d(V)^2 \overline{R}_V^{\gamma *} C_{V,\overline{V}}^* \gamma_V (\gamma_{\overline{V}} (C_{\mho,\overline{Z}} T^\gamma C_{X,Y}) C_{\overline{V},X \circ Y} d_{\overline{V},X \circ Y}^*)$$

$\boxed{E_V} C_{V,\overline{V}} \overline{R}_V^\gamma$

(res)

$$= d(\mho) \gamma_\mho (R_\mho^{\gamma *} C_{\overline{\mho},\mho}^*) \boxed{E_\mho} \gamma_\mho (\quad — // — \quad)$$

Thus

$\upsilon_X \gamma_X (\upsilon_Y) C_{X,Y} \upsilon_{X \circ Y}^*$

$$\approx \sum_{\mho, Z, T} \frac{d(\mho)^3 d(Z)}{d(X)} \overline{R}_\mho^{\gamma *} C_{\mho,\overline{\mho}}^* \gamma_\mho (C_{\mho,Y,Z,T}) E_\mho^\gamma C_{\mho,\overline{\mho}} \overline{R}_\mho^\gamma$$

$$C_{\mho,Y,Z,T} = d_{\overline{\mho},X} C_{\overline{\mho},X}^* \gamma_{\overline{\mho}} (T^{\gamma *} C_{\mho,\overline{Z}}^* \gamma_\mho (d_{\overline{Z},Y} C_{\overline{Z},Y}^*))$$

$$\cdot C_{\overline{\mho},\mho} R_\mho^\gamma R_\mho^{\gamma *} C_{\overline{\mho},\mho}$$

$$\cdot \gamma_{\overline{\mho}} (C_{\mho,\overline{Z}} T^\gamma C_{X,Y}) C_{\overline{\mho},X \circ Y} d_{\overline{\mho},X \circ Y}^*.$$