§1.1 Basic Extensions

Let $Q \subset P$ vN algs $(1_Q = 1_P)$

$$\begin{tikzcd} Q \subset P \arrow[d, squiggly] \\ (L^2(P),\ L^2(P)_+,\ J) \end{tikzcd}$$

$(L^2(P), L^2(P)_+, J)$ std. Hilb. sp. of P

Positive cone (for $L^2(P)_+$), modular conj (for J)

- $J\xi = \xi$ $\forall \xi \in L^2P_+$
- $\forall \eta \in L^2P$ & $\langle \eta, \xi \rangle \geqq 0$ $\forall \xi \in L^2P_+$ $\Rightarrow \eta \in L^2P_+$

$J : L^2P \to L^2P$ conj. linear isometric $J^2 = 1$.

$$\boxed{JPJ = P'}$$

$$\begin{tikzcd} P' \subset Q' \arrow[d, "J\cdot J"] & \subset B(L^2P) \\ JP'J \subset JQ'J \\ P \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

Defn. 1.1

For $Q \subset P$, we say $P \subset JQ'J$ is the <u>basic extension</u>. ($JQ'J$ often denoted by P_1)

☆ std. Hilb. sp unique $\rightsquigarrow$ $P \subset JQ'J$ determined by $Q \subset P$.

If we want to describe $JQ'J$ explicitely, we use an approppreate std Hilb. sp.

Suppose $\exists\ E : P \to Q$ f.n. cond. exp.

(We often denote by $Q \overset{E}{\subset} P$ this situation).

Take $\varphi \in Q_*^+$ faithful.

$\to \varphi \circ E \in P_*^+$ faithful

$\to L^2P := L^2(P, \varphi\circ E)$ $\leftarrow$ the GNS sp.

$= \overline{P\xi_{\varphi\circ E}}$

$J := J_{\varphi\circ E}$ the modular conj.

$L^2P_+ := \overline{\{xJxJ\,\xi_{\varphi\cdot E} \mid x \in P\}}$

(std. form)

Let

$$e_Q : L^2(P, \varphi\cdot E) \to \overline{Q\,\xi_{\varphi\cdot E}}$$

the ortho. proj. (the Jones Proj)

Lem. 1.2

e_Q satisfies the following.

(1) $e_Q \in Q'$

$\overset{4}{(\cancel{2})}$ $Je_Q = e_Q J$

$\overset{2}{(\cancel{3})}$ $e_Q\, x\, \xi_{\varphi\cdot E} = E(x)\,\xi_{\varphi\cdot E} \quad \forall x \in P$

$\overset{3}{(\cancel{4})}$ $e_Q\, x\, e_Q = E(x)\, e_Q \quad \forall x \in P$

(5) $Q = \{e_Q\}' \cap P$

Proof.

(1) Trivial ($\overline{Q\,\xi_{\varphi\cdot E}}$ is $\overset{*\text{-alg}}{\downarrow}{Q}$-inv.)

(2) $e_Q^{\circ}\, x\, \xi_{\varphi\cdot E} := E(x)\,\xi_{\varphi\cdot E} \quad (x \in P)$

(well-defn. since $\xi_{\varphi\cdot E}$ sep. vector)

$\| e_Q^{\circ} \| \leqq 1$?

$$\| e_Q^{\circ}\, x\, \xi_{\varphi\cdot E} \|^2 = \| E(x)\,\xi_{\varphi\cdot E} \|^2$$
$$= \varphi\cdot E(E(x)^* E(x))$$

Kadison ineq.

$E(x)^*E(x) \leqq E(x^*x)$ $\to$ $\leqq \varphi\cdot E(E(x^* x))$

$$= \varphi\cdot E(x^* x)$$
$$= \| x\,\xi_{\varphi\cdot E} \|^2 \quad \rightsquigarrow \| e_Q^{\circ} \| \leqq 1.$$

e_Q° extends to L^2P

$\rightarrow$ $e_Q^{\circ} \in B(L^2P)$.

~~Trivial~~ idemp. : $e_Q^{\circ} \cdot e_Q^{\circ} = e_Q^{\circ}$ ~~idemp.~~ triv.

Self-adj? $e_Q^{\circ *} = e_Q^{\circ}$

$$(\because)\ \langle e_Q^{\circ *}\, x\,\xi_{\varphi\cdot E},\ y\,\xi_{\varphi\cdot E} \rangle = \langle x\,\xi_{\varphi\cdot E},\ E(y)\,\xi_{\varphi\cdot E} \rangle$$
$$\| $$
$$\varphi\cdot E(E(y)^* x) \quad (E \text{ cp map})$$
$$\| $$
$$\varphi E(E(y^*) x)$$
$$\| $$
$$\varphi(E(y^*) E(x))$$
$$\| \quad \text{symmetricity}$$
$$\langle e_Q^{\circ}\, x\,\xi_{\varphi\cdot E},\ y\,\xi_{\varphi\cdot E} \rangle \quad \blacksquare$$

Thus $e_Q^{\circ} = e_Q$ on L^2P.

(3) $e_Q \overset{\in P}{x} e_Q \overset{\in P}{y} \xi_{\varphi.E}$

$= e_Q x E(y) \xi_{\varphi.E}$

$= E(x E(y)) \xi_{\varphi.E}$

$= E(x) E(y) \xi_{\varphi.E}$

$= E(x) e_Q y \xi_{\varphi.E}$.

(4) (Takesaki)

Let S be the Tomita's S:

$$S x \xi_{\varphi.E} := x^* \xi_{\varphi.E} \quad (x \in P)$$

$\left(\begin{array}{l} S : \text{conj. linear. closed op. on } L^2P \\ S^2 \subset 1. \\ P\xi_{\varphi.E} \underset{\substack{\uparrow \\ \text{core inclusion.}}}{\subset} \mathcal{D}(S) \qquad S = J\Delta_{\varphi.E}^{1/2} \end{array} \right)$

Then

$S e_Q x \xi_{\varphi.E} = S E(x) \xi_{\varphi.E}$

$= E(x)^* \xi_{\varphi.E}$

$= e_Q x^* \xi_{\varphi.E}$

$= e_Q S x \xi_{\varphi.E}$. $\quad {}^\forall x \in P$

$\to e_Q S \subset S e_Q$.

$\rightsquigarrow (1-2e_Q) S = S \underset{\substack{\uparrow \\ \text{unitary}}}{(1-2e_Q)}$

$\rightsquigarrow (1-2e_Q) \Delta_{\varphi.E} = \Delta_{\varphi.E} (1-2e_Q)$

$(1-2e_Q) J_{\varphi.E} = J_{\varphi.E} (1-2e_Q)$

$\rightsquigarrow \begin{cases} e_Q \Delta_{\varphi.E}^{it} = \Delta_{\varphi.E}^{it} e_Q \\ e_Q J_{\varphi.E} = J_{\varphi.E} e_Q \end{cases}$

☆ In particular, this implies $\sigma_t^{\varphi.E}(x) = \sigma_t^{\varphi}(x)$
$(x \in Q)$

In fact. for

$$\begin{tikzcd} Q \arrow[r, phantom, "\subset"] & P \arrow[dl, "\psi"] \\ \mathbb{C} & \end{tikzcd}$$

f.n. pos. TFAE

(1) ${}^\exists E : P \to Q$ f.n. cond. exp s.t. $\psi.E = \psi$

(2) $\sigma_t^{\psi}(Q) = Q \quad {}^\forall t \in \mathbb{R}$.

(5) By (1), $Q \subset \{e_Q\}' \cap P$.

Let $x \in \{e_Q\}' \cap P$.

Then $E(x) e_Q = e_Q x e_Q = x e_Q$

$\rightsquigarrow E(x) \xi_{\varphi.E} = E(x) e_Q \xi_{\varphi.E} = x e_Q \xi_{\varphi.E} = x \xi_{\varphi.E}$

$\to \overset{\in Q}{E(x)} = x$ ∎

Lem. 1.3

$$J_{\varphi.E}\,\overset{\substack{\text{the basic ext. of } Q\subset P\\ \downarrow}}{Q'}\,J_{\varphi.E} = P \vee \{e_Q\}''$$

Proof.

By Lem1.2(5), $Q = \{e_Q\}' \cap P'$

$$\xrightarrow{\text{comm}} Q' = \{e_Q\}'' \vee P'$$

$$\xrightarrow{J\cdot J} JQ'J = \{Je_QJ\}'' \vee JP'J$$

$$= \{e_Q\}'' \vee P$$

∎

Then

$$Z(P_1) = J_{\varphi.E}\, Z(Q')\, J_{\varphi.E}$$

$$= J\, Z(Q) J$$

i.e.

$$\begin{tikzcd} Z(Q) \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & Z(P_1) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ a \arrow[r, mapsto] & Ja^*J \end{tikzcd}$$

vN. isomo.

Lem. 1.4

$\forall z \in Z(P_1)$ $\exists ! z_0 \in Z(Q)$ s.t. $z = Jz_0J$

Moreover, characterized by $z e_Q = z_0^* e_Q$.

Proof.

$$z e_Q \underset{\substack{\uparrow\\ P}}{x}\xi_{\varphi.E} = Jz_0J\, E(x)\xi_{\varphi.E}$$

$x \in D(\sigma_{i/2}^{\varphi.E})$ $$= J z_0\, E(\sigma_{-i/2}^{\varphi.E}(x))^* \xi_{\varphi.E}$$

$$= J E(\sigma_{i/2}^{\varphi.E}(\underset{z_0^* x}{\cancel{x z_0}})^*)\xi_{\varphi.E}$$

$$= E(z_0^* x)\xi_{\varphi.E}$$

$$= z_0^* e_Q x \xi_{\varphi.E}$$

∎

§1.2 Exp with finite index

Let
$$Q \overset{E}{\subset} P \subset \underset{\underset{JQ'J = P \vee \{e_Q\}''}{\parallel}}{P_1}$$

Lem. 1.5

$$P e_Q P := \mathrm{span}\{x e_Q y \mid x, y \in P\}$$

is a σ-weakly dense $*$-~~closed ideal~~ subalg. in P_1 ⌋

Proof.

- $x e_Q y \cdot a e_Q b = x E(ya) e_Q b$.
- $(x e_Q y)^* = y^* e_Q x^*$ $\rightarrow$ $*$-subalg

$\cdot\ \overline{P e_Q P}^{\sigma\text{-}w} \subset P_1$

$\uparrow$

vN alg.

NOTE: $\underset{\underset{I}{\parallel}}{\overline{P e_Q P}^{\sigma\text{-}w}}$ ideal in P_1

(∵) Trivial. ~~P~~ $xI + Ix \subset I$. $\forall x \in P$

$e_Q \cdot a e_Q b = E(a) e_Q b$

$a e_Q b e_Q = a E(b) e_Q$

$\exists z \in P_1$ central proj s.t.

$$\overline{P e_Q P}^{\sigma\text{-}w} = P_1 z.$$

Take $z_0 \in Z(Q)$ s.t. $z = J z_0 J$.

Then $\quad e_Q \in P_1 z$?

$$\underset{\underset{Q \subset P}{\cap}}{z_0^*} e_Q = z e_Q \overset{\checkmark}{=} e_Q$$

$\longrightarrow z_0^* = 1$

$\longrightarrow z = 1$

☆ Analogy :

$$F(H) = \overset{\text{span}}{\{}\xi \overset{\text{shatten form}}{\odot} \eta \mid \xi, \eta \in H\}$$

$\updownarrow$

$x e_Q y$

Defn. 1.6

$Q \overset{E}{\subset} P$.

E is of <u>fin. index</u>

if $\quad P_1 = P e_Q P$

⌟

Lem. 1.7

$Q \overset{E}{\subset} P$. TFAE.

(1) E fin index.

(2) $\exists a_k \in P \quad k = 1 \cdots n$. s.t.

$$\sum_{k=1}^{n} a_k e_Q a_k^* = 1$$

proof.

(2) ⇒ (1) trivial

(1) ⇒ (2) Take b_l, c_l s.t.

$$\sum_l b_l e_Q b_l^* = 1$$

Polarization

$\rightsquigarrow \exists b_m', c_m'$

$$\sum_m b_m' e_Q b_m'^* - \sum_m c_m' e_Q c_m'^* = 1.$$

$$\to \quad 1 \leq \underset{\substack{\uparrow \\ \text{invertible}}}{\sum_m b_m' e_Q b_m'^*} =: h$$

$$\rightsquigarrow 1 \neq h^{-\frac{1}{2}} \sum_m b_m' e_Q b_m'^* h^{-1/2}$$

$$\shortparallel$$

$$\sum_m (\underset{\substack{\cap \\ P_1}}{h^{-\frac{1}{2}} b_m'}) e_Q (h^{-\frac{1}{2}} \cdot b_m')^*$$

Since $P_1 e_Q = P e_Q$, $\exists a_m \in P$

$a_m e_Q = h^{-\frac{1}{2}} b_m'$

$$\to \quad \sum_m a_m e_Q a_m^* = 1.$$

▨

Defn. 1.8

$\{a_k\}_{k=1}^n$ is a <u>quasi-base</u> of E

$\overset{\text{defn}}{\Longleftrightarrow}$ · $a_k \in P$

· $\displaystyle\sum_{k=1}^{n} a_k e_Q a_k^* = 1$

⌟

<u>NOTE.</u>

$$\sum_k a_k e_Q a_k^* = 1 \iff \sum_k a_k E(a_k^* x) = x \quad \forall x \in P$$

$$\iff \sum E(x a_k) a_k^* = x \quad \forall x \in P.$$

§1.3 Dual Operator Valued Wt & Index of E

Suppose $Q \overset{E}{\subset} P$ fin. index.

Let $\{a_k\}_k$ quasi-base of E.

Then ~~we have~~ consider the map $\underset{\text{c.p.}}{}$

$$\begin{tikzcd} Q' \arrow[r] & P' \\ x \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & {} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\begin{tikzcd} B(L^2P) \arrow[r] & B(L^2P) \\ x \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \sum_{k=1}^{n} a_k x a_k^* \arrow[u, phantom, "\cup"] \end{tikzcd}$$

☆ If $x \in Q'$, then $\sum a_k x a_k^* \in P'$

(∵) Let $y \in P$. then

$$\sum_k a_\ell x \underbrace{a_k^* y} = \sum_{k,\ell} a_k \overset{\in Q'}{x} \boxed{\overset{\in Q}{E(\underbrace{a_k^* y}\, a_\ell)}}\, a_\ell^* \quad \text{QB.}$$

$$= \sum_{k,\ell} a_k E(a_k^* y a_\ell) x a_\ell^*$$

$$= y \sum_\ell a_\ell x a_\ell^*$$

∎

Actually, this map does not dep. on $\{a_k\}$.

Indeed, let $\{b_\ell\}_{\ell=1}^{m}$ another Q.B. of $Q \overset{E}{\subset} P$.

Then for $x \in Q'$,

$$\sum a_k x a_k^* = \sum_{k,\ell} b_\ell \overset{\in Q}{E(b_\ell^* a_k)} \cdot \overset{\in Q'}{x} a_k^*$$

$$= \sum_{k,\ell} b_\ell x E(b_\ell^* a_k) a_k^*$$

$$= \sum_\ell b_\ell x b_\ell^*$$

Defn. 1.9

For $Q \overset{E}{\subset} P$ fin. index, we set

$$\begin{tikzcd} E^{-1}: Q' \arrow[r] & P' \\ x \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \sum_{k=1}^{n} a_k x a_k^* \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where $\{a_k\}$ Q.B. of E.

☆ E^{-1} = Haagerup's E^{-1}

Thm.(H) $M \supset N$

$$\begin{tikzcd} P(M,N) \arrow[r, leftrightarrow] & P(N',M') \\ T \arrow[u, phantom, "\cup"] \arrow[r, leftrightarrow] & T^{-1} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$$\frac{d\varphi\cdot T}{d\psi'} = \frac{d\varphi}{d\psi'\cdot T^{-1}}$$

Lem. 1.10

$E^{-1} : Q' \to P'$ satisfies the follwng.

(1) E^{-1} is P'-bimodular

(2) $E^{-1}(e_Q) = 1$

⌟

Proof.

(1) $E^{-1}(\underset{P'}{a}\overset{Q'}{x}\underset{P'}{b}) = \sum \underset{P}{a_k}\, axb\, a_k^* = aE^{-1}(x)\, b$

(2) $E^{-1}(\underset{Q'}{e_Q}) = \sum a_k e_Q a_k^* = 1.$ ▨

☆ E^{-1} is characterized by (1) & (2)

(∵) $P_1 = Pe_Q P$
$\quad \| $
$JQ'J$

$\rightsquigarrow Q' = P' e_Q P'.$

since

$E^{-1}(1) \geqq E^{-1}(e_Q) \geqq 1.$

$E^{-1}(1)$ invertible &

$\underset{P'}{a} E^{-1}(1) = E^{-1}(a) = E^{-1}(1)\, a$

$\rightsquigarrow E^{-1}(1) \in Z(P)$

Defn. 1.11

$Q \overset{E}{\subset} P$ finite index The index of E:

$\operatorname{Ind} E := E^{-1}(1) \in Z(P).$

⌟

☆ $\operatorname{Ind} E \geqq 1.$

$\operatorname{Ind} E = \sum_k a_k a_k^*$

$\uparrow$ Q.B.

Ex. 1.12 $\mathrm{id}\otimes\varphi$

$$Q \subset Q\otimes M_n(\mathbb{C}) \qquad \varphi = \mathrm{Tr}(h\,\cdot\,)$$

$$h = \sum_{k=1}^{n} \lambda_k e_{kk}$$

$$e_{k\ell}\, e_Q\, e_{k\ell}^* \,(a\otimes e_{\underset{rs}{\cancel{m\ell}}})\xi_\varphi$$

$$= e_{k\ell}\, E(a\otimes e_{\ell s})\cdot \delta_{k,r}\, \xi_\varphi$$

$$= e_{k\ell}\cdot a\otimes \varphi(e_{\ell s})\,\delta_{k,r}\, \xi_\varphi$$

$$= a\otimes \delta_{kr}\,\delta_{\ell s}\,\lambda_\ell\, e_{k\ell}\, \xi_\varphi$$

$$\rightsquigarrow \sum_{k,\ell} \sqrt{\lambda_\ell}^{-1} e_{k\ell}\; e_Q\; (\sqrt{\lambda_\ell}^{-1} e_{k\ell})^* = 1.$$

$$\rightsquigarrow \mathrm{Ind}\,E_h = \sum_{k\ell} \sqrt{\lambda_\ell}^{-1} e_{k\ell}\cdot(\sqrt{\lambda_\ell}^{-1} e_{k\ell})^*$$

$$= \sum_{k\ell} \sqrt{\lambda_\ell}^{-2}\cdot e_{kk}$$

$$= \mathrm{Tr}(h^{-1})\cdot 1$$

NOTE.

$$\min_{\substack{h \\ \text{pos inv.} \\ \mathrm{Tr}(h)=1}} \mathrm{Ind}\,E_h = n^2$$

$$\uparrow$$

$$h = 1.$$

Ex. 1.13

$$G \overset{\alpha}{\curvearrowright} M \quad \rightsquigarrow \quad M \overset{E}{\subset} M\rtimes_\alpha G$$

finite grp $\qquad$ vN alg

$$\| $$

$$\operatorname{span}_{s\in G} M\lambda^\alpha(s)$$

$$E(\lambda^\alpha(s)) = \delta_{s,e}$$

$\{\lambda^\alpha(s)\}_s$ Q.B.

(∵) Check

$$x = \sum \lambda^\alpha(s)\, E(\lambda^\alpha(s)^*\cdot x)$$

or

$$x = \sum E(x\lambda^\alpha(s)^*)\,\lambda^\alpha(s)$$

$$x = a\lambda^\alpha(t) \to E(x\lambda^\alpha(s)^*) = E(a\lambda^\alpha(ts^{-1}))$$

$$= a\,\delta_{ts^{-1},e}$$

$$= a\,\delta_{t,s}$$

$$\rightsquigarrow \text{RHS} = \sum_s \delta_{s,t}\, a\lambda^\alpha(s) = a\lambda^\alpha(t) = x \quad \blacksquare$$

$$\mathrm{Ind}\,E = \sum_s \lambda^\alpha(s)\lambda^\alpha(s)^* = \sum_s 1 = |G|$$

Set

$$\begin{tikzcd} P_1 \arrow[r, "E_1"] \arrow[d, "J\cdot J"'] & P \\ Q' \arrow[r, "E^{-1}"] & P' \arrow[u, "J\cdot J"'] \end{tikzcd} \quad \times \frac{1}{\mathrm{Ind}\,E}$$

i.e.

$$E_1(x) = J E^{-1}(JxJ)J \cdot \frac{1}{\mathrm{Ind}\,E}$$

$$x \in P_1$$

$\rightsquigarrow$ $E_1 : P_1 \to P$ cond. exp. & $E_1(e_Q) = (\mathrm{Ind}\,E)^{-1}$

☆ $\{a_k\}_k$: Q.B. of E

$\Rightarrow$ $\{\underset{\underset{b_k}{\shortparallel}}{a_k e_Q} (\mathrm{Ind}\,E)^{\frac{1}{2}}\}_k$: Q.B. of E_1

(∵)

$$\sum_k b_k E_1(b_k^* \overset{\in P}{x} e_Q \overset{\in P}{y})$$

$$= \sum_k b_k \; E_1(\underset{\underset{Z(P)}{\cap}}{(\mathrm{Ind}\,E)^{\frac{1}{2}}} \underbrace{e_Q a_k^* x e_Q y}_{\overset{\shortparallel}{e_Q E_Q(a_k^* x) y}})$$

$$= \sum_k b_k \cdot (\mathrm{Ind}\,E)^{\frac{1}{2}} (\mathrm{Ind}\,E)^{-1} E_Q(a_k^* x) y$$

$$= \sum_k \underset{\underset{\text{Q.B.}}{\uparrow}}{a_k} e_Q E_Q(a_k^* x) y = x e_Q y .$$

$$\mathrm{Ind}\,E_1 = \sum_k a_k e_Q (\mathrm{Ind}\,E)^{\cancel{-1}} e_Q a_k^*$$

$$= \sum_k a_k e_Q a_k^* \; \mathrm{Ind}\,E$$

if $\mathrm{Ind}\,E \in \mathbb{R}$

$$= \mathrm{Ind}\,E .$$

§1.4 Pimsner - Popa ineq.

$Q \overset{E}{\subset} P$ fin. index, a_k : Q.B. fixed. $k = 1 \cdots \underline{n}$

Lem. 1.14

Let
$$\begin{tikzcd} P \arrow[r, "\pi"] & Q \otimes M_n(\mathbb{C}) \\ x \arrow[u, phantom, "\in" rotate] \arrow[r, mapsto] & {[E(a_k^* x a_l)]_{k.l}} \arrow[u, phantom, "\in" rotate] \end{tikzcd}$$

(1) π is a faithful normal $*$-homo.

(2) $\pi(1) \underset{\neq}{\geq}$ the supp proj of $[a_k^* a_l]$ in $P \otimes M_n(\mathbb{C})$

Proof.

(1) $\sum_l E(a_k^* x a_l) E(a_l^* y a_m)$. $\quad x, y \in P$

$= E(a_k^* x \sum_l a_l E(a_l^* y a_m))$

$= E(a_k^* x y a_m)$. $\quad \rightarrow$ multiplicative.

If $\pi(x) = 0$ then

$e_Q a_k^* x a_l e_Q = 0 \quad \rightarrow x = 0.$

$\rightsquigarrow \sum_{k.l} a_k \underset{\downarrow 1}{e_Q a_k^*} x a_l \underset{\downarrow 1}{e_Q a_l^*} = 0$

(2) ~~$\pi(1)$~~ $[a$

$\sum_l E(a_k^* a_l) \cdot a_l^* a_m = a_k^* a_m$

$\rightsquigarrow \pi(1) \ [a_k^* a_l]_{k.l} = [a_k^* a_l]_{k.l}$

$$\begin{tikzcd} Q \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & P \arrow[r, phantom, "<"] \arrow[d, "\pi"] & P_1 \\ \pi(Q) \arrow[r, phantom, "<"] & \pi(P) & \end{tikzcd}$$

Applying $P \overset{E_1}{\subset} P_1 \quad b_k := a_k e_Q (\mathrm{Ind}\, E)^{\frac{1}{2}}$

$$\begin{tikzcd} P_1 \arrow[r, "\pi_1"] & P \otimes M_n(\mathbb{C}) \\ x \arrow[u, phantom, "\in" rotate] \arrow[r, mapsto] & {[E_1(b_k^* x b_l)]_{k.l}} \arrow[u, phantom, "\in" rotate] \end{tikzcd} \quad *\text{-homo.}$$

$\pi_1(\overset{\overset{P}{\cup}}{x}) = [E_1((\mathrm{Ind}\, E)^{\frac{1}{2}} e_Q \underset{\overset{\|}{E(a_k^* x a_l) e_Q}}{a_k^* x a_l e_Q} (\mathrm{Ind}\, E)^{\frac{1}{2}})]$

$= \pi(x)$

$\pi_1(e_Q) = [E_1((\mathrm{Ind}\, E)^{\frac{1}{2}} E(a_k^*) E(a_l) e_Q (\mathrm{Ind}\, E)^{\frac{1}{2}})]$

$= [E(a_k^*) E(a_l)]$ $\quad \cdots$

Lem. 1.15 (PP ineq)

Let $Q \overset{E}{\subset} P$ fin ind.

Then $\exists\, c > 0$ s.t. $E(x) \geqq cx \quad {}^\forall x \in P_+$ ⌟

Proof.

$a_k \in P \quad k = 1 \cdots n \quad$ Q. B. of E.

$$P \ni {}^\forall x = \sum_k a_k E(a_k^* x)$$

$$x^* x = \sum_{k.\ell} E(x^* a_k)\, a_k^*\, a_\ell\, E(\overset{x}{a_\ell^* a_k})$$

$$E(x^* x) = \sum_{k.\ell} E(x^* a_k)\, E(a_k^* a_\ell)\, E(a_\ell^* x)$$

$$= \underset{\text{row}}{[E(x^* a_k)]_k}\ \underbrace{[E(a_k^* a_\ell)]_{k\ell}}_{\uparrow\ \pi(1)}\ \underset{\text{column}}{[E(a_\ell^* x)]_\ell}$$

$$\geqq c\, [E(x^* a_k)]_k\, [a_k^* a_\ell]\, [E(a_\ell^* x)]_\ell$$

$$= c\, x^* x.$$

where $c := \| [a_k^* a_\ell] \|^{-1}$ works

$= \| \sum a_k a_k^* \|^{-1}$

$= \| \mathrm{Ind}\, E \|^{-1}$

▤

Prop. 1.16

$Q \overset{E}{\subset} P$ T. F. A. E.

(1) E fin. index.

(2) $\exists\, c > 0.\quad E(x) \geqq cx \quad \forall x \in P_+$ ⌟

Proof.

(1) ⇒ (2) The prev Lem.

(2) ⇒ (1) we will prove this for

(i) $Q \subset \underset{\nwarrow \text{ fin. dim}}{P}$

(ii) $\underset{\substack{\uparrow \\ \text{prop. inf vN alg}}}{Q} \subset P$

(i) $Q \subset P \subset P_1 \subset \underset{\substack{\uparrow \\ \text{fin dim}}}{B(L^2 P)}$

$\rightarrow P_1 = \overline{P e_Q P}^w = P e_Q P.$ ok.

(ii')

<u>Claim 1.</u> $e_Q \in P_1$ prop. inf proj.

(∵) $e_Q P_1 e_Q = \overline{e_Q P e_Q P e_Q}^w$

$= \overline{Q e_Q}^w$

$= \underset{\substack{\uparrow \\ \text{vN alg.}}}{Q}\, e_Q \cong Q$

▤

<u>Claim 2.</u> $P_1 e_Q = P e_Q$

(∵) $P_1 e_Q = \overline{P e_Q P e_Q}^{w} = \overline{P e_Q}^{w} \supset P e_Q$

Let $x \in P_1$. Take a net $y_\lambda \in P$ s.t.

- $x e_Q \xleftarrow{\sigma\text{-}w} y_\lambda e_Q$
- $\|y_\lambda e_Q\| \leqq \|x e_Q\|$ Kaplanski

$$\rightsquigarrow \|x\|^2 \geqq \|x e_Q\|^2 \geqq \|y_\lambda e_Q\|^2$$
$$= \|e_Q y_\lambda^* y_\lambda e_Q\|$$
$$= \|E(y_\lambda^* y_\lambda) e_Q\| \qquad \left(\begin{array}{ccc} Q & \to & Q e_Q \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & x e_Q \end{array}\right)$$
$$= \|E(y_\lambda^* y_\lambda)\|$$
$$\overset{PP}{\geqq} c \, \|y_\lambda^* y_\lambda\|$$

Thus $\|y_\lambda\| \leqq c^{-\frac{1}{2}} \|x\| \; (\forall \lambda)$
bdd net

$y_\lambda \xrightarrow[\sigma\text{-}w]{\text{subnet}} y \in P$

Clearly $x e_Q - y e_Q$ ∎

Since $z_{P_1}(e_Q) = 1$ & e_Q prop inf in P_1
Claim 1.

$\exists\, w \in P_1$ s.t. $e_Q = w^* w$, $1 = w w^*$.

Take $a \in P$ s.t.

$w = w e_Q = a e_Q$ by Claim 2

Then $1 = a e_Q a^*$.

i.e. $\{a\}$ is a Q.B. of E ∎

<u>Rem. 1.17</u>

☆ In the proof. we have shown,

for $\underset{\substack{\uparrow \\ \text{prop inf}}}{Q} \overset{E}{\subset} P$ fin index

$\exists\, a \in P$ s.t.

- $a e_Q a^* = 1$. (i.e. a is a Q.B.)
- $E(a^* a) = 1$.

In particular, $\mathrm{Ind} E = a a^*$.

§ 1.5 Abstract characterizations of Basic Ext.

Prop. 1.18

For $Q \overset{E}{\subset} P \subset R$ (inclusions of vNalgs),

suppose the following :

(i) $\mathrm{Ind}\,E < \infty$

(ii) $R = P \vee \{ \underset{\uparrow}{e} \}''$
projection with $Z_P(e) = 1$.

(iii) $exe = E(x)e \quad \forall x \in P$.

Then $\exists\, \theta : P_1 \longrightarrow R$ $*$-isomo s.t.

- $\theta(x) = x \quad \forall x \in P$
- $\theta(e_Q) = e$

┘

Proof.

Claim.1 $R = \overline{PeP}^{w}$

(∵) $\underset{\uparrow}{\overline{PeP}^{w}} = Rz \qquad z \in R$ central proj
ideal by
(ii) & (iii)

$ze = e \underset{(ii)}{\rightsquigarrow} z = 1$

C_1 ▨

Claim 2 $Re = Pe$

(∵) $Re = \overline{PePe}^{w} = \overline{Pe}^{w} \supset Pe$

By using the PP ineq, we can show $\subset$ as before.

C_2 ▨

Claim 3. $a_k \in P$ $(k=1 \cdots n)$ Q. B. of E

$\Rightarrow \sum_{k=1}^{n} a_k e a_k^* = 1_R$

(∵) For $x, y \in P$

$\underset{\uparrow}{\sum_{k=1}^{n} a_k e a_k^*} \cdot xey = \sum_{k=1}^{n} a_k E(a_k^* x) ey$
unit of R $= xey$

C_3 ▨

Now let

$\theta : P_1 \to R$

$\sum \underset{\in P}{x_r} e_Q \underset{\in P}{y_r} \longmapsto \sum x_r e y_r$

well-defined ?

$0 = \sum x_r e_Q y_r \Leftrightarrow 0 = \sum_{r,s} x_r E(y_r y_s^*) e_Q x_s^* \qquad \hat{E}(e_Q)=(\mathrm{Ind}E$

$\overset{\hat{E}}{\Longleftrightarrow} 0 = \sum x_r E(y_r y_s^*) x_s^*$

$= \underset{\text{row}}{(x_r)} (\underset{\text{pos}}{E(y_r y_s^*)})_{r,s} (x_s^*)^*$

$\geq (x_r) (\qquad)_{r,s} (e \otimes 1)(x_s^*)$

$= \sum_{r,s} x_r E(y_r y_s^*) e x_s^*$

$\Leftarrow \sum x_r e y_r = 0$.

P ▨

Prop. 1.19 (when we don't know $\mathrm{Ind}\,E < +\infty$)

For $Q \overset{E}{\subset} P \overset{F}{\subset} R$,

suppose

- $R = P \vee \{e\}''$ ($\uparrow$ proj with $z_R(e) = 1$)
- $exe = E(x)\,e \quad \forall x \in P$
- $F(e) \in Z(P)$ invertible.

Then $\exists\, \theta : P_1 \to R$ $*$-isomo s.t.

- $\theta(x) = x \quad \forall x \in P$
- $\theta(e_Q) = e$
- $\theta \circ \hat{E}_1 \circ \theta^{-1} = F$

$\lrcorner$

Proof.

For $x \in P$,

$$\|E(x^*x)\| = \|E(x^*x)e\| = \|xe\|^2$$

$$= \|xex^*\| \geqq \|F(xex^*)\|$$

$$= \|xF(e)x^*\| \geqq c\,\|xx^*\|. \qquad c>0 \text{ s.t. } F(e) \geqq c1.$$

(norm) PPineq.

$\rightsquigarrow Re = Pe \ \&\ \mathrm{Ind}\,E < +\infty$

a_k : Q.B. of E

$$\rightsquigarrow \sum_k a_k e a_k^* = 1$$

$$\overset{F}{\rightsquigarrow} \sum_k a_k F(e) a_k^* = 1.$$

$$\parallel$$

$$\sum_k a_k a_k^* \cdot F(e) \qquad \to F(e) = (\mathrm{Ind}\,E)^{-1} = \overset{\cancel{\wedge}}{E_1}(e)$$

∎

Ex. 1.20

$G \overset{\alpha}{\curvearrowright} M$.

fin grp, vN alg.

$$\begin{tikzcd} M \arrow[r, phantom, "\overset{E}{\subset}"] & M\rtimes_\alpha\Gamma \arrow[r, phantom, "\overset{(F)}{\subset}"] & M\otimes B(\ell^2\Gamma) \arrow[d, phantom, "\ni"] & R \arrow[l] \\ Q \arrow[u] & P \arrow[u] & e := 1\otimes e_{1,1} & \end{tikzcd}$$

($1 \in \Gamma$ unit)

- $\pi_\alpha(x)\,e = x \otimes e_{11}$.
- $\lambda^\alpha(s)\,e\,\lambda^\alpha(t)^* = 1 \otimes e_{st}$

$\Big] \rightsquigarrow R = P \vee \{e\}''$.

$z(e) = 1\otimes 1$ trivial

- $\underline{e\,\pi_\alpha(x)}\,\lambda^\alpha(t)\,e = \underline{(x\otimes e_{11})} \cdot \overset{\alpha}{\cancel{\lambda(t)}}\,(1\otimes e_{t1}) = \delta_{t,1}\, x\otimes e_{11}$

$$= E(\pi_\alpha(x)\,\lambda^\alpha(t))\,e$$

By prop.$^{1.18}$, $P_1 \simeq R \quad e_Q \mapsto e$.

Ex. 1.21

$$\underset{\substack{\uparrow\\ \text{fin gre}}}{G} \overset{\alpha}{\curvearrowright} \underset{\text{factor}}{M} \qquad \left(\begin{array}{l} \alpha \text{ is } \underline{\underline{\text{outer}}} \\ \quad \downarrow \\ \quad M' \cap (M \rtimes_\alpha \Gamma) = \mathbb{C} \end{array} \right)$$

$$M^\alpha \overset{E}{\subset} M \overset{F}{\subset} M \rtimes_\alpha \Gamma \qquad \text{such that } \overset{\downarrow}{M \rtimes_\alpha \Gamma} \text{ factor.}$$

$$\begin{cases} E(x) = \sum_{s \in G} \alpha_s(x) \cdot \frac{1}{|G|}, \quad x \in M \\ F\left(\sum \pi_\alpha(x(s)) \overset{\alpha}{\lambda}(s) \right) := \pi_\alpha(x(e)). \end{cases}$$

$$e := \frac{1}{|G|} \sum_{s \in G} \overset{\alpha}{\lambda}(s) \in M \rtimes_\alpha \Gamma$$

$$\uparrow$$

$$\text{Proj. } \quad z(e) = 1 \leftarrow z(M \rtimes_\alpha \Gamma) = \mathbb{C}1 ? \quad \swarrow \text{outer.}$$

- $$e\, \pi_\alpha(x) \underset{\widehat{\cancel{\lambda(t)}}}{\overset{\alpha}{\cancel{e}}} = \sum_{s.t.} \overset{\alpha}{\lambda}(s)\, \pi_\alpha(x)\, \overset{\alpha}{\lambda}(\cancel{r}t)\ |G|^{-2}$$

$$e\, \pi_\alpha(x) e = \sum_{s.t} \pi_\alpha(\alpha_s(x))\, \overset{\alpha}{\lambda}(s\cancel{r}t)\ |G|^{-2} \qquad t \to (s\cancel{r})^{-1} t$$

$$= \pi_\alpha(E(x))\, e.$$

$$\rightsquigarrow \underset{\text{ideal}}{\overline{MeM}^{w}} \underset{\substack{\uparrow\\ \text{factor}}}{=} M \rtimes_\alpha \Gamma$$

- $$F(e) = \frac{1}{|G|} \quad \overset{\text{Prop 1.19}}{\rightsquigarrow} \quad \operatorname{Ind} E = \overset{M_1 \simeq M \rtimes_\alpha \Gamma \ \downarrow}{\cancel{\frac{1}{|G|}}} |G| = \operatorname{Ind} F$$

$\boxed{\text{済}}$

§ 1.6 cond. exp & rel. comm.

For $Q \subset P$,

$$\mathcal{E}(P, Q) := \{ E \mid E : P \to Q \text{ f.n. cond. exp} \}$$

Lem. 1.22

Suppose $Q \overset{E}{\subset} P$ fin. index.

Then $\forall F \in \mathcal{E}(P, Q)$ $\exists ! h \in Q' \cap P$ s.t.

$$F(x) = E(hx) \quad \forall x \in P$$

Proof.

~~Injectivity~~ (uniqueness) From E faithful.

(Existence).

$\{a_k\}_k$ Q.B. of E.

$$F(\underset{\underset{P}{\Uparrow}}{x}) = F\left(\sum_k a_k E(a_k^* x)\right)$$

$$= \sum_k F(a_k) E(a_k^* x)$$

$$= E\left(\sum_k F(a_k) a_k^* x\right).$$

Put $h := \sum_k F(a_k) a_k^*$

We show $h \in Q' \cap P$.

For ${}^\forall y \in Q$,

$$y h = \sum_k F(y a_k) a_k^*$$

$$= \sum_{k, l} F(a_l E(a_l^* y a_k)) a_k^*$$

$$= \sum_{k, l} F(a_l) E(a_l^* y \underline{\underline{a_k}}) \underline{\underline{a_k^*}}$$

$$= \sum_l F(a_l) a_l^* y$$

$$= h y.$$

☒

$*$ cf. Known fact:

$$\begin{tikzcd} \mathcal{E}(P, Q) \arrow[r] & \mathcal{E}(Q' \cap P, Z(Q)) \\ E \arrow[u, phantom, "\in" description] \arrow[r, mapsto] & E\restriction_{Q' \cap P} \arrow[u, phantom, "\in" description] \end{tikzcd}$$

If $\mathcal{E}(P, Q) \neq \phi$, $\uparrow$ is bijective.

Section 2 C^*-tensor categories

§2.1 Quick review of C^*-tensor cat.

$\mathcal{C}$: a category is a C^*-tensor cat

if

(1) $\forall X, Y \in \mathcal{C}$, $\mathcal{C}(X,Y)$ Banach sp.

$$\begin{array}{ccc} \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) & \longrightarrow & \mathcal{C}(X,Z) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (S,T) & \longmapsto & TS \end{array}$$

bilinear & $\|TS\| \leqq \|T\|\|S\|$.

(2) $\exists * : \mathcal{C}(X,Y) \to \mathcal{C}(Y,X)$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ T & \longleftrightarrow & T^* \end{array}$$

conj. linear. map s.t.

- $T^{**} = T$
- $\|T^*T\| = \|T\|^2$
- $T^*T \in \mathcal{C}(X,X)_+$

((1) & (2) → we call C^*-category).

<u>NOTE</u> $\mathcal{C}(X,X)$ unital C^*-alg.

(3) $\exists$

$- \otimes - : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$

bilinear bifunctor.

$(X,Y) \longmapsto X \otimes Y$.

$$\left.\begin{array}{l} S \in \mathcal{C}(X,Y) \\ T \in \mathcal{C}(U,V) \end{array}\right. \mapsto S \otimes T \in \mathcal{C}(X \otimes U, Y \otimes V)$$

bilinear.

$\exists \underset{\substack{X,Y,Z \\ \uparrow\ \uparrow\ \uparrow}}{a} : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$

natural in X, Y, Z. unitary morphism. (called the associator)

s.t.

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes U \arrow[dl, "a_{X,Y,Z} \otimes 1_U"'] \arrow[dr, "a_{X \otimes Y, Z, U}"] & \\
(X \otimes (Y \otimes Z)) \otimes U \arrow[d, "a_{X, Y \otimes Z, U}"'] & \circlearrowleft & (X \otimes Y) \otimes (Z \otimes U) \arrow[d, "a_{X,Y,(Z \otimes U)}"] \\
X \otimes ((Y \otimes Z) \otimes U) \arrow[rr, "1_X \otimes a_{Y,Z,U}"] & & X \otimes (Y \otimes (Z \otimes U))
\end{tikzcd}$$

(4) $\exists\, \underset{\substack{\uparrow \\ \text{tensor unit.}}}{\mathbb{1}} \in \mathcal{C}$ $\quad \exists\, \underset{\uparrow \text{natural}}{\lambda_X} : \mathbb{1} \otimes X \to X$ unitary

$\exists\, \underset{\uparrow \text{natural.}}{\rho_X} : X \otimes \mathbb{1} \to X$ unitary

s.t.

- $$\begin{tikzcd} (X \otimes \mathbb{1}) \otimes Y \arrow[rr, "a_{X,\mathbb{1},Y}"] \arrow[dr, "\rho_X \otimes 1_Y"'] & \circlearrowleft & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "1_X \otimes \lambda_Y"] \\ & X \otimes Y & \end{tikzcd}$$

- $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$.

(5) $(S \otimes T)^* = S^* \otimes T^*$

(6) $\mathrm{End}(\mathbb{1}) = \mathbb{C}$.

(7) $\forall X, \forall Y \in \mathcal{C}$. $\exists Z \in \mathcal{C}$. $\exists S \in \mathcal{C}(X, Z)$

$T \in \mathcal{C}(Y, Z)$

isometries

s.t.

$$SS^* + TT^* = 1_Z$$

(In this case we denote $Z = X \oplus Y$).

(8) $\forall X \in \mathcal{C}$ $\forall p \in \mathrm{End}(X)$ proj.

$\exists Y \in \mathcal{C}$. $\exists S \in \mathcal{C}(Y, X)$ isometry

s.t. $SS^* = p$.

☆ (8) is sometimes dropped.

It is known that

If $\mathcal{C}$ with (1) ~ (7),

unique upto equivalence $\to$ ⓔ $\widetilde{\mathcal{C}}$ with (1) ~ (8)

s.t. $\mathcal{C} \longrightarrow \widetilde{\mathcal{C}}$ fully faithful C^*-tensor functor.

Karoubi envelope (idempotent completion) . $\forall X \in \widetilde{\mathcal{C}}$, $\exists Y \in \mathcal{C}$ $\exists p \in \mathcal{C}(Y, Y)$

s.t. $\exists S \in \widetilde{\mathcal{C}}(Y, X)$

$SS^* = p$.

☆ If $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ equal obj

$a_{X,Y,Z} = 1$. trivial associators

$\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$

$\lambda_X = 1_X = \rho_X$

then $\mathcal{C}$ is <u>strict</u>. (Always assumed here)

Defn. 2.1

$\mathcal{C}$: C^*-$\otimes$ cat.

$X, Y \in \mathcal{C}$ are <u>conj. pair</u>

if $\exists S \in \mathcal{C}(\mathbb{1}, Y \otimes X)$

$\exists \bar{S} \in \mathcal{C}(\mathbb{1}, X \otimes Y)$

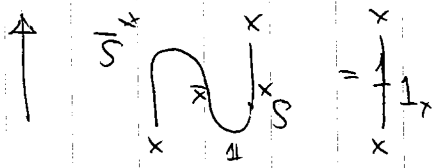
s.t.

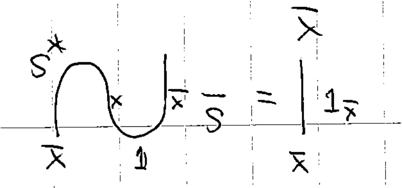
conjugate equation*

$$\begin{cases} (\bar{S}^* \otimes 1_X)(1_X \otimes S) = 1_X \\ (S^* \otimes 1_Y)(1_Y \otimes \bar{S}) = 1_Y. \end{cases}$$

Graphically

direction of morphisms





☆ (X, Y), (X, Z) conj. pairs $\Rightarrow Y \simeq Z$.

☆ $d^{(S,\bar{S})}(X) := \|S\| \|\bar{S}\|$

unique up to "unitary equiv" → ⓔ $(R, \bar{R})$ sol. of conj eq of (X, Y) $\|R\| = \|\bar{R}\|$.

s.t.

$$d^{(R,\bar{R})}(X) = \min_{(S,\bar{S})} d^{(S,\bar{S})}(X)$$

called the std. sol.

$\perp$

We will write $d(X) := d^{(R,\bar{R})}(X) \geq 1$.

the dimension of X.

☆ $\mathcal{C}$ is <u>rigid</u> if $\forall X \in \mathcal{C}$. $\exists Y \in \mathcal{C}$

s.t. (X, Y) conj. pair.

$d(X \oplus Y) = d(X) + d(Y)$

$d(X \otimes Y) = d(X)\, d(Y)$

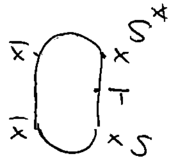
$d(\mathbb{1}) = 1$

$d(\bar{X}) = d(X)$

☆ Let $(S, \bar{S})$ sol of conj eq for $X, \underset{\bar{X}}{\underset{\shortparallel}{Y}}$.

$$\varphi_X^{(S,\bar{S})} : \underset{\underset{T}{\rotatebox{90}{$\in$}}}{\mathrm{End}(X)} \to \underset{\rotatebox{90}{$\in$}}{\mathbb{C}}$$

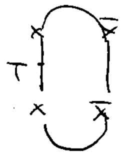
$$T \longmapsto$$



$$= S^*(1_Y \cdot T)S$$

$$\psi_X^{(S,\bar{S})} : \underset{\underset{T}{\rotatebox{90}{$\in$}}}{\mathrm{End}(X)} \to \mathbb{C}$$

$$T \longmapsto$$



$$= \bar{S}^*(T \cdot 1_{\bar{X}})\bar{S}.$$

Then $(S, \bar{S})$ std $\Longleftrightarrow$ $\varphi_X^{(S,\bar{S})} = \psi_X^{(S,\bar{S})}$.

Furthermore

- If $(S, \bar{S})$ std, then $\varphi_X^{(S,\bar{S})} = \psi_X^{(S,\bar{S})}$ tracial.
- cf $(S, \bar{S})$, $(S', \bar{S}')$ std. then

$$\varphi_X^{(S,\bar{S})} = \varphi_X^{(S',\bar{S}')}$$

$\lrcorner$

§2.2 C^*-2-categories

Λ : index set.

$\mathcal{C}_{rs}$: C^*-category $r, s \in \Lambda$

$-\otimes- : \mathcal{C}_{r\underline{\underline{s}}} \times \mathcal{C}_{\underline{\underline{s}}t} \to \mathcal{C}_{rt}$ bilinear functor.

obj $(X, Y) \mapsto X \otimes Y$

mor $(T, S) \mapsto T \otimes S$

$1\!\!1_s$: tensor unit in $\mathcal{C}_{ss}$

$1\!\!1_s \otimes X = X$ $X \in \mathcal{C}_{st}$

$Y \otimes 1\!\!1_s = Y$ $Y \in \mathcal{C}_{rs}$.

etc.

$\mathcal{C} := (\mathcal{C}_{rs})_{r,s \in \Lambda}$ is a <u>C^*-2-category</u>.

For $X \in \mathcal{C}_{rs}$ & $Y \in \mathcal{C}_{sr}$, we say they are <u>conjugates</u> if $\mathcal{C}_{rr}(1\!\!1_r, X \otimes Y)$ containing $\mathcal{C}_{ss}(1\!\!1_s, Y \otimes X)$

a solution of conj. eq.

☆ $\mathcal{C}^\circ_{rs}$:= obj $X \in \mathcal{C}_{rs}$ with conjugate.

morphism $= \mathcal{C}_{rs}(X, Y)$.

→ $\mathcal{C}^\circ := (\mathcal{C}^\circ_{rs})_{r,s}$ full subcategory.

$\underline{\underline{\text{rigid}}}$

($\forall X$ has a conjugate)

☆ We will treat $\Lambda = \{0, 1\}$, the 2pt set.

For A, B : unital C^*-algs (or inf factors),

$Mor(A, B) := \{\pi \mid \pi : A \to B$ unital faithful $*$-homo$\}$

(if A, B vN alg, π assumed to be normal.)

objs $\rho, \sigma \in Mor(A, B)$

mor $(\rho, \sigma) := \{t \in B \mid t\rho(x) = \sigma(x)t \ \forall x \in A\}$

$\rightsquigarrow$ $Mor(A, B)$ C^*-category.

Putting $A_0 := A$, $A_1 := B$.

$\mathcal{C}_{00} := \mathrm{Mor}(A_0, A_0)$ $\quad$ $\mathcal{C}_{01} := \mathrm{Mor}(A_1, A_0)$

$\mathcal{C}_{10} := \mathrm{Mor}(A_0, A_1)$ $\quad$ $\mathcal{C}_{11} := \mathrm{Mor}(A_1, A_1)$,

& bilinear functor

$$\begin{tikzcd}
\mathcal{C}_{rs} \times \mathcal{C}_{st} \arrow[r] & \mathcal{C}_{rt} \\
(\rho, \sigma) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \rho\sigma \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

$\rightsquigarrow$ $\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \mathcal{C}_{10} & \mathcal{C}_{11} \end{pmatrix}$ $\quad$ C^*-2-category. $\quad$ ← direct sum はもんだい.

$\mathrm{Mor}(A_s, A_r)_0 := \{\rho \in \mathrm{Mor}(A_s, A_r) \mid \text{with conj}\}$

$$\mathrm{Mor}(A_s, A_r)_0 = \mathcal{C}_{rs}^0 \subset \mathcal{C}_{rs}$$

$\rightsquigarrow$ $\mathcal{C}^0 := \begin{pmatrix} \mathcal{C}_{00}^0 & \mathcal{C}_{01}^0 \\ \mathcal{C}_{10}^0 & \mathcal{C}_{11} \end{pmatrix}$ $\quad$ rigid C^*-2-category.

§2.3. categorical index & minimal index

Thm.2.2

N, M; inf factors

$\rho \in \mathrm{Mor}(N, M)$. TFAE.

(1) $\rho \in \mathrm{Mor}(N, M)_0$

(2) $^\exists E : M \to \rho(N)$ cond. exp. fin index faithful. normal $\lrcorner$

Proof.

(1) ⇒ (2) Let $\sigma \in \mathrm{Mor}(M, N)_0$ a conj of ρ.

Take

$S \in (\mathrm{id}_N, \sigma\rho) \subset N$

$\bar{S} \in (\mathrm{id}_B, \rho\sigma) \subset M$ a solution of conj eq.

Let

$$E(x) := \rho(S^*\sigma(x)S) \cdot \lambda \qquad \lambda := \frac{1}{\|S^*S\|}$$

#

— (*)

Claim. $E : M \to \rho(N)$ cond. exp.

(∵) $\|E\| \leq 1$ trivial

$$E(\rho(x)) = \rho(S^* \sigma\rho(x) S)\lambda$$
$$= \rho(x \underline{S^*S})\underline{\lambda}$$
$$= \rho(x)$$

証

faithful

$$0 = E(x^*x) \Leftarrow \rho(\sigma(x)^*S) = 0$$
$$\Longrightarrow \bar{S}^* \rho\sigma(x)\rho(S) = 0$$
$$\parallel$$
$$x\,\bar{S}^*\rho(S) = x$$

証

Claim $\left(M = \bar{S}^*\rho(N)\right)$ or. $\left\{\frac{\bar{S}^*}{\sqrt{\lambda}}\right\}$ Q.B of E.

(∵)

$$\frac{1}{\lambda}\bar{S}^* E(\bar{S}x) \qquad x \in M$$
$$= \frac{1}{\cancel{\lambda}}\bar{S}^*\rho(\underbrace{S^*\sigma(\bar{S}}_{1}x)S)\cdot\cancel{\lambda}$$
$$= \bar{S}^*\rho(\sigma(x)S)$$
$$= x\,\bar{S}^*\rho(S) = x$$

証

$$\rightsquigarrow \mathrm{Ind}\,E = \frac{\bar{S}^*}{\sqrt{\lambda}} \cdot \frac{\bar{S}}{\sqrt{\lambda}} = \frac{\bar{S}^*\bar{S}}{\lambda} = \|S\|^2\|\bar{S}\|^2 = d^{(S,\bar{S})}(\rho)^2$$

(1)⇒(2) 証

$(2) \Rightarrow (1)$

Since N infinite factor & $\rho(N) \overset{E}{\subset} M$ finite index

$\exists\, a$ ~~S~~ $\in M$ Q.B. of E.

$a\, e_{\rho(N)}\, a^* = 1$

$E(a^* a) = 1.$ Rem 1.17

~~Recall~~

We set $\bar{S} := \sqrt{\lambda}\, a^* \in M$ + $\lambda = (\mathrm{Ind}E)^{-1/2}$

i.e. $\bar{S}^* e_{\rho(N)} \bar{S} = \lambda$ $\rightsquigarrow$ $\bar{S}^* \bar{S} = \mathrm{Ind}E^{\frac{1}{2}} = \lambda^{-1}$

$E(\bar{S}\bar{S}^*) = \lambda$

Recall Lem 1.14,

$$\begin{tikzcd}
M \arrow[r, "\pi"] & \rho(N) \otimes M_1(\mathbb{C}) \arrow[r, "\rho^{-1}"] & N \\[-15pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-15pt]
x \arrow[r, mapsto] & E(a^* x a) \arrow[r, mapsto] & \rho^{-1} E(a^* x a) \\[-15pt]
 & & \rotatebox{90}{$=$} \\[-15pt]
 & & \sigma(x)
\end{tikzcd}$$

i.e. $\sigma(x) := \rho^{-1} E(\bar{S} x \bar{S}^*) \frac{1}{\lambda}$ $x \in M$.

~~We~~

We will show (ρ, σ) is a conj. pair.

$\bar{S} \in (\mathrm{id}_M, \rho\sigma)$

$(\because)$

$$\rho\sigma(x)\bar{S} = \frac{1}{\lambda} E(\bar{S} x \bar{S}^*)\bar{S}$$

$$= E(\bar{S} x a) a^*$$

$$= \bar{S} x \qquad \text{Q.B.}$$

∎

We construct $S \in (\mathrm{id}_N, \sigma\rho)$

Look at $(1) \Rightarrow (2)$ $(*)$

$E(\bar{S}) = \lambda\, \rho(S^* \sigma(\bar{S}) S) = \lambda \rho(S)$

$$S := \lambda^{-1} \rho^{-1} E(\bar{S})$$

$$S^* S = \lambda^{-2} \rho^{-1}(E(\bar{S}^*) E(\bar{S}))$$

$$= \lambda^{-1} \rho^{-1}(E(a E(a^*)))$$

$$= \lambda^{-1}.$$

$S \in (\mathrm{id}_N, \sigma\rho)$

$(\because)$ $\underset{N \ni}{S x} = \lambda^{-1} \rho^{-1}(E(\bar{S})\rho(x)) = \lambda^{-1} \rho^{-1}(E(\bar{S}\rho(x)))$

$$= \lambda^{-1} \rho^{-1}(E(\rho\sigma\rho(x)\bar{S})) = \sigma\rho(x) S$$

We now check the conj. eq.:

$$S^*\sigma(\bar S) = \lambda^{-1}\rho^{-1}(E(\bar S^*)\rho\sigma(\bar S))$$
$$= \lambda^{-1}\rho^{-1}(E(\bar S^*\rho\sigma(\bar S)))$$
$$= \lambda^{-1}\rho^{-1}(E(\bar S\bar S^*))$$
$$= 1$$

$$\bar S^*\rho(S) = \bar S^*\cdot\lambda^{-1}E(\bar S)$$
$$= a\,E(a^*)$$
$$= 1$$

∎

<u>Rem 2.3</u>

Lem 1.14 gives an explicite ~~formula~~ of $\bar\rho$.

Now let.

$\rho : N \to M$ with conj. $\sigma : M \to N$

$\mathcal{S} := \{(S,\bar S) \mid$ solves the conj. eq.$\}$

From the proof of Thm 2.2 (1) ⇒ (2)

$$\begin{tikzcd}
\mathcal{S} \arrow[r] & \mathcal{E}(M,\rho(N)) \arrow[r, "\mathrm{Ind}"] & \mathbb{R} \\
(S,\bar S) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \rho(S^*\sigma(\cdot)S)\cdot\frac{1}{\|S\|^2} \arrow[u, phantom, sloped, "\in"] \arrow[d, phantom, sloped, "="] & \\
 & E^{(S,\bar S)} \arrow[r, mapsto] & \|S\|^2\|\bar S\|^2 = \overset{S,\bar S}{d}(\rho)^2
\end{tikzcd}$$

We show the surjectivity.

Take the std. sol. $(R,\bar R)$ $\qquad \|R\| = \|\bar R\| = d(\rho)^{\frac{1}{2}}$

i.e. $R^*(1\otimes t)R = \bar R^*(t\cdot 1)\bar R^*$ $\qquad t \in (\rho,\rho)$ trace.

tensor cat. notation

$$\underset{\substack{\shortparallel \\ \mathrm{Tr}_\rho(t)}}{R^*\sigma(t)R} = \bar R^* t\bar R \qquad R^*R = d(\rho)$$

$$\rightsquigarrow \quad E^{(R,\bar R)}\restriction_{(\rho,\rho)} = \frac{1}{d(\rho)}\mathrm{Tr}\rho(\cdot) \quad \text{tracial state.}$$

Now. let us take $F \in \mathcal{E}(M,\rho(N))$.

Lem 1.22 $\rightsquigarrow$ $\exists!\, h \in \rho(N)'\cap M = (\rho,\rho)$ s.t.

$$F(\cdot) = E^{(R,\bar R)}(h\cdot)$$

Since $F\Gamma_{(\rho,\rho)}$ state & $E^{(R,\bar{R})}\Gamma$ tracial st.

we have $h \geq 0$, and $F = E^{(R,\bar{R})}(h^{\frac{1}{2}} \cdot h^{\frac{1}{2}})$
invertible

$$\left(\because \; F\Gamma_{(\rho,\rho)} = E(h^{\frac{1}{2}} \cdot h^{\frac{1}{2}})\Gamma_{(\rho,\rho)} \right)$$

Expectation.

i.e.

$$F(x) = \frac{1}{d(\rho)}\, \rho(R^* \sigma(h^{\frac{1}{2}} x\, h^{\frac{1}{2}}) R) \qquad F(1) = \frac{Tr_\rho(h)}{d(\rho)}$$

~~$E(h^{-\frac{1}{2}}$~~

$(\sigma(h^{\frac{1}{2}})R,\ h^{-\frac{1}{2}}\bar{R})$ solves the conj eq.

$\|\sigma(h^{\frac{1}{2}})R\|^2 = Tr_\rho(h) = d(\rho)$

& $F = E^{(\ ,\ )}$ (i.e. surjectivity)

$$\begin{aligned} \mathrm{Ind}\, F &= \|\sigma(h)^{\frac{1}{2}} R\|^2 \, \|h^{-\frac{1}{2}}\bar{R}\|^2 \\ &= Tr_\rho(h)\ \bar{R}^* h^{-1} \bar{R} \\ &= d(\rho)\ Tr_\rho(h^{-1}) \qquad \geq Tr_\rho(1)^2 = d(\rho)^2 \end{aligned}$$

Prop. 2.4

$\rho \in \mathrm{Mor}(N, M)_0$

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[dr, "d"'] & \mathcal{E}(M, \rho(N)) \arrow[d, "\mathrm{Ind}"] \\ & \mathbb{R} \end{tikzcd}$$

G

In particular

$$\min_{E} \mathrm{Ind} E = d(\rho)^2$$

NOTATION

$E_\rho(x) = \rho(R_\rho^* \sigma(x) R_\rho)$, $x \in M$

↑ std isometry

indep

$\phi_\rho(x) = R_\rho^* \sigma(x) R_\rho$ left inverse.

$\phi_\rho : M \to N$

$E_\sigma(x) = \sigma(\bar{R}_\rho^* \rho(x) \bar{R}_\rho)$ $x \in N$

$\phi_\sigma(x) = \bar{R}_\rho^* \rho(x) \bar{R}_\rho \quad : N \to M$

Section 3 Subfactors & C^*-2-cats.

§3.1. From a subfactor to a C^*-2-category.

Let

$\rho \in \mathrm{Mor}(N, M)_0$ ← N, M infinite factors.

$\bar\rho \in \mathrm{Mor}(M, N)_0$ a conj of ρ.

We set

$$\mathcal{C}^\rho := \begin{pmatrix} \mathcal{C}^\rho_{00} & \mathcal{C}^\rho_{01} \\ \mathcal{C}^\rho_{10} & \mathcal{C}^\rho_{11} \end{pmatrix}$$

$$\underset{\nwarrow}{\subset} \begin{pmatrix} \mathrm{Mor}(N,N)_0 & \mathrm{Mor}(M,N)_0 \\ \mathrm{Mor}(N,M)_0 & \mathrm{Mor}(M,M)_0 \end{pmatrix}$$

full subcat.

$\Lambda = \{N, M\}$
$\quad\ \ \updownarrow\ \updownarrow$
$\quad = \{0, 1\}$

defined as follows:

- $\mathcal{C}^\rho_{00}$ ($N-N$ sectors)

Obj = all direct summands of $(\bar\rho\rho)^n$, $n \geq 0$
+ isomorphic objects NOTE $\mathrm{id}_N \in \mathcal{C}^\rho_{00}$

- $\mathcal{C}^\rho_{10}$ ($M-N$ sectors)

Obj = all direct summands of $\rho(\bar\rho\rho)^n$ $n \geq 0$
+ isomorphic objects NOTE $\rho \in \mathcal{C}^\rho_{10}$

- $\mathcal{C}^\rho_{01}$ ($N-M$ sectors)

Obj = ~~$\bar\rho(\rho\bar\rho)^n$~~ all direct summands of $\bar\rho(\rho\bar\rho)^n$ $n \geq 0$
+ isomorphic objects NOTE $\bar\rho \in \mathcal{C}^\rho_{01}$

- $\mathcal{C}^\rho_{11}$ ($M-M$ sectors)

Obj = all direct summands of $(\rho\bar\rho)^n$ $n \geq 0$.
+ isomorphic objects NOTE $\mathrm{id}_M \in \mathcal{C}^\rho_{11}$.

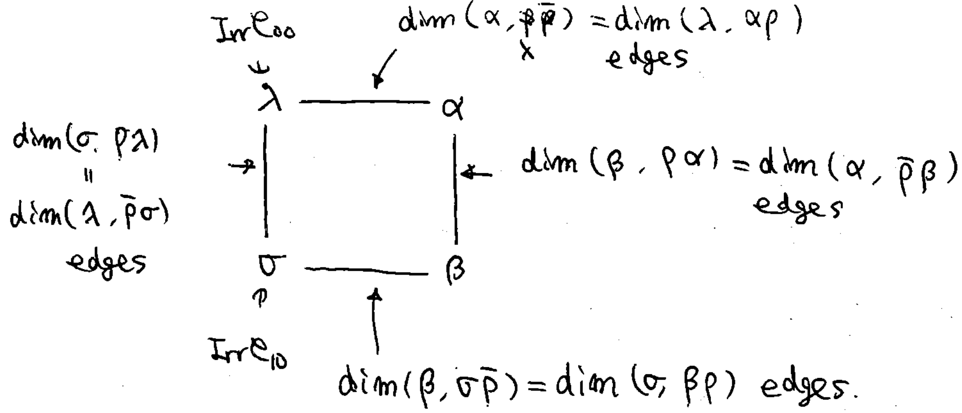
☆
$$\begin{tikzcd} \mathcal{C}^\rho_{00} \arrow[r, bend left, "\otimes\bar\rho"] \arrow[d, bend right, "\rho\otimes\cdot"'] & \mathcal{C}^\rho_{01} \arrow[l, bend left, "\otimes\rho"] \arrow[d, bend right, "\rho\otimes"'] \\ \mathcal{C}^\rho_{10} \arrow[r, bend left, "\otimes\bar\rho"] \arrow[u, bend right, "\bar\rho\otimes\mathrm{id}"'] & \mathcal{C}^\rho_{11} \arrow[l, bend left, "\otimes\rho"] \arrow[u, bend right, "\bar\rho\otimes\cdot"'] \end{tikzcd}$$

⇝ 4 graphs are associated.

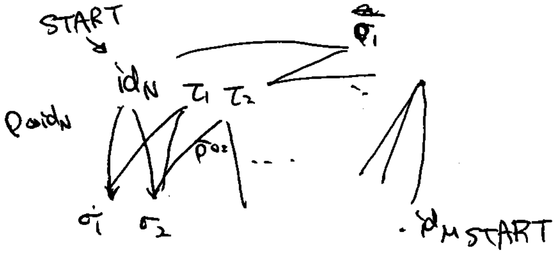
Vertex sets = $\mathrm{Irr}\,\mathcal{C}^\rho_{00} \sqcup \mathrm{Irr}\,\mathcal{C}^\rho_{10} \sqcup \mathrm{Irr}\,\mathcal{C}^\rho_{01} \sqcup \mathrm{Irr}\,\mathcal{C}^\rho_{11}$.

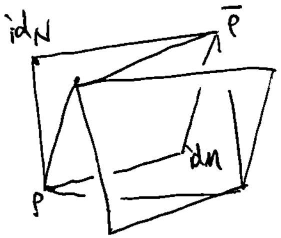
$\lambda \underset{\uparrow}{\overline{\qquad}} \sigma$ multiplicity of σ in $\rho\lambda$ or $\lambda\rho$ or $\bar\rho\lambda$ or $\lambda\bar\rho$.

unoriented edges.



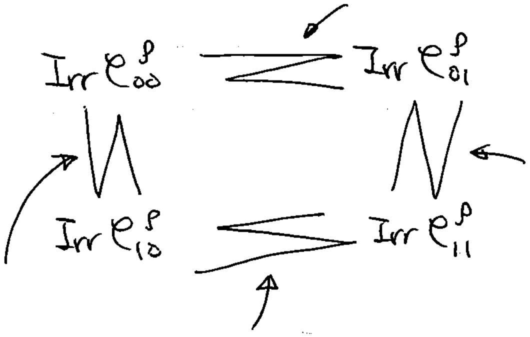
To draw the graph, Step-by-Step method is also useful:



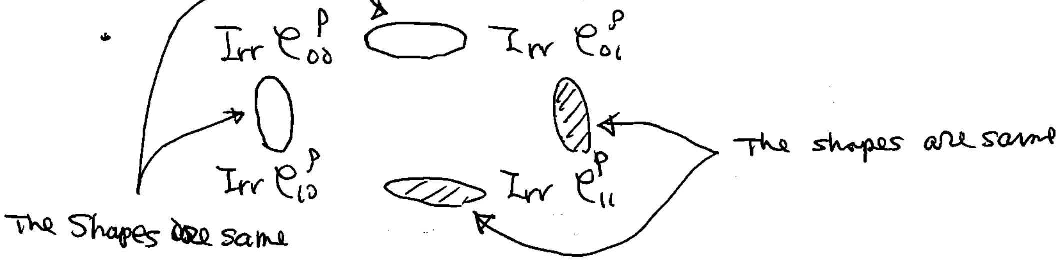


Example of A4 graphs

Rem. 3.1.

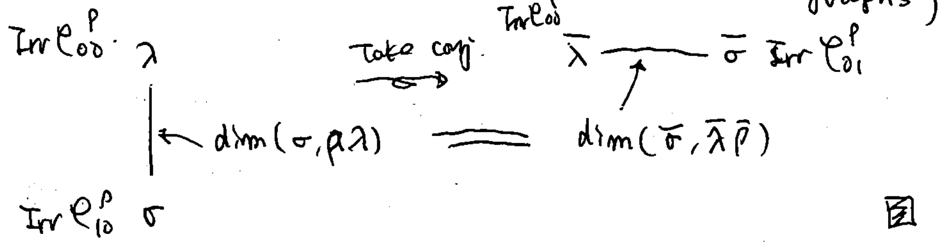
- The graph of $\mathcal{C}^\rho$ is connected.
  obtained by tensoring $\rho\otimes\cdot$, $\bar\rho\otimes\cdot$, $\cdot\otimes\rho$ and $\cdot\otimes\bar\rho$.
  & decompositions
- 

  Each graph is bipartite.

- 

  i.e. Essentially 2 graphs. (Principal / Dual principal graphs)

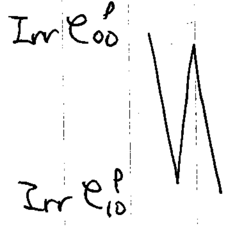
(∵)



☒

Next consider the adjacency matrices.

For example,



graph

$\rightsquigarrow$ $\Lambda = (\Lambda_{\lambda\sigma})$: $\mathrm{Irr}\, e^{\rho}_{00} \times \mathrm{Irr}\, e^{\rho}_{10}$ -matrix. (possibly ∞ size)

with $\Lambda_{\lambda\sigma} := \#$ edges of $\lambda - \sigma$
$= \dim(\sigma, \rho\lambda)$

Let

$$\vec{d}_{rs} := (d(\alpha))_{\alpha \in \mathrm{Irr}\, e_{rs}} \quad \forall r, s \in \{0,1\}$$

Then

$$d(\rho)\, \vec{d}_{00} = \Lambda \vec{d}_{10}, \quad d(\rho)\, \vec{d}_{10} = {}^t\Lambda\, \vec{d}_{00}.$$

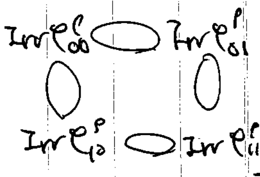
(∵)
$$\begin{aligned}(\Lambda \vec{d}_{10})(\lambda) &= \sum \Lambda_{\lambda\sigma}\, d(\sigma) \\ &= \sum \dim(\sigma, \rho\lambda)\, d(\sigma) \\ &= d(\rho\lambda) \\ &= d(\rho)\, d(\lambda)\end{aligned}$$ ▨

Lem. 3.2

$$\|\Lambda\| \leqq d(\rho)$$

$\uparrow$

the adjacency matrix of one of



Proof.

By Schwz test :

If $A = (a_{k,l})_{k \in K,\, l \in L}$

s.t.

- $a_{kl} \geqq 0 \quad \forall k, l$
- $\exists \vec{v} = (v_k)_k, \ \vec{w} = (w_l)_l$
  $\exists \alpha, \exists \beta > 0$
  with
  - $v_k > 0, \ w_l > 0$
  - $A \vec{w} \leqq \alpha \vec{v}$
  - $A^t \vec{v} \leqq \beta \vec{w}$

then $A \in B(\ell^2 L, \ell^2 K)$ & $\|A\| \leqq \sqrt{\alpha\beta}$ ▨

Set

$$\dim \mathcal{C}_{rs}^{\rho} := \sum_{\lambda \in \mathrm{Irr}\,\mathcal{C}_{rs}} d(\lambda)^2 \qquad r,s \in \{0,1\}$$

the global dim of $\mathcal{C}_{rs}^{\rho}$.

Prop 3.3.

$$\dim \mathcal{C}_{00}^{\rho} = \dim \mathcal{C}_{10}^{\rho} = \dim \mathcal{C}_{01}^{\rho} = \dim \mathcal{C}_{11}^{\rho}$$ $\lrcorner$

Proof.

Λ : $\mathrm{Irr}\,\mathcal{C}_{10}^{\rho} \times \mathrm{Irr}\,\mathcal{C}_{00}^{\rho}$ adj. matrix

$$\begin{array}{c}\mathrm{Irr}\,\mathcal{C}_{00}^{\rho}\\ \bigvee\\ \mathrm{Irr}\,\mathcal{C}_{10}^{\rho}\end{array}$$

Then

$$\begin{array}{ccc}
\langle \Lambda^t \Lambda \vec{d_{10}}, \vec{d_{10}} \rangle & = & d(\rho) \langle \Lambda^t \vec{d_{00}}, \vec{d_{10}} \rangle \\
\| & & \| \\
\langle \Lambda \vec{d_{10}}, \Lambda \vec{d_{10}} \rangle & & d(\rho)^2 \langle \vec{d_{10}}, \vec{d_{10}} \rangle \\
\| & & \| \\
d(\rho)^2 \langle \vec{d_{00}}, \vec{d_{00}} \rangle & & d(\rho)^2 \dim \mathcal{C}_{10}^{\rho} \\
\| & & \\
d(\rho)^2 \dim \mathcal{C}_{00}^{\rho} & &
\end{array}$$

▨

☆ Two C^*-$\otimes$ cats (rigid) $\mathcal{C}$ & $\mathcal{D}$ are Morita equiv

when $\exists$ rigid C^*-2-cat $\begin{bmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \underset{\neq 0}{\mathcal{C}_{10}} & \mathcal{C}_{11} \end{bmatrix}$

s.t.

$$\mathcal{C} \underset{\otimes}{\cong} \mathcal{C}_{00}, \quad \mathcal{D} \underset{\otimes}{\cong} \mathcal{C}_{11}.$$

Hence if $\mathcal{C} \underset{\text{Morita}}{\cong} \mathcal{D}$, then $\dim \mathcal{C} = \dim \mathcal{D}$.

☆ When $\# \mathrm{Irr}\,\mathcal{C}_{00}^{\rho} < \infty$, in Lem 3.2,

$$\| \Lambda \| = d(\rho)$$

($\because$) $\vec{d_{rs}}$ are Perron-Frob. eigenvectors

▨

Ex. 3.24

fin. grp.

$$N \xrightarrow{\rho} N \rtimes_{\alpha} G =: M$$

inf. factor　　outer action

$((\alpha_s, \alpha_t) = \{0\}$ if $s \neq t \Leftrightarrow \rho(N)' \cap M = \mathbb{C})$

$\rho(x) := \pi_\alpha(x) \quad x \in N$

Let $E_\rho : M \to \rho(N)$

$\sum \rho(x_t) \lambda^\alpha(t) \mapsto \rho(x_e)$

Lem 1.22

Then $\mathcal{E}(M, \rho(N)) = \{E_\rho\}$.

We know $\mathrm{Ind}\, E_\rho = |G|$　Ex. 1.13

Q.B. $\{\lambda^\alpha(t)\}_{t \in G}$.

$d(\rho) = |G|^{\frac{1}{2}}$

We want to describe

$$e^\rho = \begin{pmatrix} e^\rho_{00} & e^\rho_{01} \\ e^\rho_{10} & e^\rho_{11} \end{pmatrix}$$

Recall $\exists a \in M$ s.t. $\{a\}$ is a Q.B. of E_ρ.

Rem 1.17　　We characterize

Let $a := \sum_{t \in G} \lambda^\alpha(t) \rho(V_t^*)$

Then

$$\begin{cases} x = a E_\rho(a^* x) & {}^\forall x \in M \quad (\text{Q.B.}) \\ E_\rho(a^* a) = 1 \end{cases}$$

NOTE
$M = \sum_t \lambda^\alpha(t) \rho(N)$

$$\Longleftrightarrow \begin{cases} \lambda^\alpha(t) = a E_\rho(a^* \lambda^\alpha(t)) & {}^\forall t \in G. & \text{—①} \\ E_\rho(a^* a) = 1. & & \text{—②} \end{cases}$$

$$a^* \lambda^\alpha(t) = \sum_s \rho(V_s) \lambda^\alpha(s)^* \lambda^\alpha(t)$$

$$\xmapsto{E_\rho} \rho(V_t^*)$$

Hence

① $\Longleftrightarrow \lambda^\alpha(t) = \sum_s \lambda^\alpha(s) \rho(V_s^* V_t) \quad {}^\forall t$

$\Longleftarrow V_s^* V_t = \delta_{s,t} 1.$

② $\Longleftrightarrow 1 = E_\rho\left(\sum_{s,t} \rho(V_s) \lambda^\alpha(s^{-1}t) \rho(V_t^*)\right)$

$= \sum_s \rho(V_s V_s^*)$

Thus

$$a = \sum_{t\in G} \lambda^{\alpha}(t)\, \rho(V_t^*) \quad \text{is a Q.B. of } E_\rho$$

$\Leftrightarrow \{V_t\}_{t\in G}$ is a Cuntz isometry in N.

<u>Claim 1</u> $\exists \{V_t\}_{t\in G}$ Cuntz isom in N

s.t.

$$\alpha_s(V_t) = V_{st} \qquad \forall s,t \in G.$$

$(\because)$

$$U_s := \sum_{t\in G} V_{st}\, \alpha_s(V_t^*) \quad , \quad s \in G$$

$\uparrow$ arbitrary Cuntz isom. in N.

$\rightsquigarrow$ $U_s\, \alpha_s(U_t) = U_{st}$

i.e. U_s 1-cocycle

α outer $\rightsquigarrow$ (Connes) $\exists\, w \in N$ unitary

s.t. $w\, U_s\, \alpha_s(w^*) = 1$.

$\rightsquigarrow$ Put $W_s := w V_s$

Then

$$1 = \sum_{t\in G} W_{st}\, \alpha_s(W_t^*)$$

$\square$

$a e_N a^* = 1$

$a a^* = |G|$

Recall Lem 1.14 & Rem 2.3.

$$\begin{tikzcd} \sigma : M \arrow[r, "\pi"] & \rho(N)\otimes M_1(\mathbb{C}) \arrow[r, "\rho^{-1}"] & N \\ x \arrow[u, phantom, "\in"] \arrow[r, mapsto] & E_\rho(a^* x a) \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \rho^{-1}E_\rho(a^* x a) \arrow[u, phantom, "\in"] \end{tikzcd}$$

where

$$a := \sum_t \lambda^{\alpha}(t)\, \rho(V_{t^{-1}}^*) \qquad V_t : \text{Cuntz satisfying } \alpha_s(V_t) = V_{st}$$

$$\overset{\text{Claim 1.}}{=} \sum_t \rho(V_e^*)\, \lambda^{\alpha}(t)$$

$$= \rho(V_e^*) \cdot |G|\, e_G$$

with $e_G := \dfrac{1}{|G|} \displaystyle\sum_{t\in G} \lambda^{\alpha}(t) \in M.$

(Ex 1.21)

We know (σ, ρ) ~~are~~ is conj. pair.

$$\bar{R}_\rho := a^* / |G|^{\frac{1}{2}}$$

$$= |G|^{\frac{1}{2}}\, e_G\, \rho(V_e)$$

$$\bar{R}_\rho \bar{R}_\rho^* = |G|\, e_G\, \rho(V_e V_e^*)\, e_G \overset{\text{Claim 1}}{=} e_G$$

$$\| $$

$$|G|\, \rho(E_G(V_e V_e^*))$$

$$\frac{1}{|G|}\sum \alpha_t(V_e V_e^*) = \frac{1}{|G|}\sum V_t V_t^* = \frac{1}{|G|}$$

No.

<u>Claim 2</u> $\sigma\rho(x) = \sum_t V_t \alpha_t(x) V_t^* \quad x \in N$

$(\rightsquigarrow \quad \sigma\rho = \bigoplus_{t\in G} \alpha_t \quad)$

(∵)

$$\alpha^* \rho(x)\, a = |G|^2\, e_G\, \rho(V_e x V_e^*)\, e_G$$

$$= |G|^2 \rho(E_G(V_e x V_e^*))\, e_G$$

where $E_G(x) := \sum_t \frac{\alpha_t(x)}{|G|} \in N^\alpha$

$$E_\rho(a^* \rho(x)\, a) = |G|^{\cancel{2}} \rho(E_G(V_e x V_e^*)) \cdot \frac{1}{\cancel{|G|}}$$

$$= \rho\Big(\sum_t \alpha_t(V_e x V_e^*)\Big)$$

$$= \rho\Big(\sum_t V_t \alpha_t(x) V_t^*\Big)$$

$\boxed{\equiv}$

$$\begin{tikzcd}
\mathcal{C}_{00}^\rho \ni 1 = \mathrm{id}_N = \alpha_e \arrow[d, "\rho\otimes\cdot"] & \alpha_t \arrow[d, "\rho\otimes\cdot"] \\
\mathcal{C}_{10}^\rho \ni \rho \arrow[ru, bend right, "\sigma\otimes\cdot", "\text{Claim 2}"'] & \rho\alpha_t
\end{tikzcd}$$

irred.

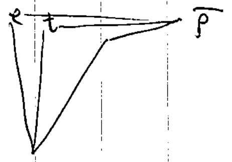
<u>NOTE.</u> $\rho\alpha_t \cong \rho$ in $\mathcal{C}_{10}^\rho$

since. $\lambda^\alpha(t)\, \rho(x) = \rho(\alpha_t(x))\, \lambda^\alpha(t)$.

i.e. $\lambda^\alpha(t) \in (\rho, \rho\alpha_t)$.

Summary :

$\mathrm{Irr}\, \mathcal{C}_{00}^\rho = \{\alpha_t\}_{t\in G} \cong G \qquad \mathrm{Irr}\, \mathcal{C}_{01}^\rho = \{\bar{\sigma}\}$



$\mathrm{Irr}\, \mathcal{C}_{10}^\rho = \{\rho\} \qquad \mathrm{Irr}\, \mathcal{C}_{11}^\rho = ?$

<u>Claim 3</u> $\sigma(M) = N^\alpha$

(∵) $E_\sigma(x) = \sigma(\bar{R}_\rho^* \rho(x) \bar{R}_\rho) \qquad x \in N \subset \rho(M)$

$$= \frac{1}{\cancel{|G|}} \rho^{-1}\big(E_\rho(a^* \bar{R}_\rho^* \rho(x) \bar{R}_\rho a)\big)$$

$$= |G|\, \rho^{-1}\big(E_\rho(\underbrace{\bar{R}_\rho \bar{R}_\rho^*}_{e_G} \rho(x) \underbrace{\bar{R}_\rho \bar{R}_\rho^*}_{e_G})\big)$$

$$= |G|\, \rho^{-1}\big(E_\rho(\rho(E_G(x))\, e_G)\big)$$

$$= E_G(x) \qquad {}^\forall x \in N$$

$\boxed{\equiv}$

Claim 4 $e^{\rho}_{11} \underset{C^*\text{-}\otimes\ \text{eq.}}{\cong} \mathrm{Rep}(G)$

(∵)

$$\begin{tikzcd} e^{\rho}_{11} \arrow[r, "F"] & \mathrm{Rep}(G) \\ \nu \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & ((\sigma, \sigma\nu), \alpha) \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

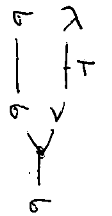
$\uparrow$

G-module

since $\alpha_t \cdot \sigma = \sigma$.

$T : \nu \to \lambda \longmapsto F(T) := \sigma(T) \in N^{\alpha}$

intertwiner



$F(T)^* = F(T)$.

$\mathrm{Mor}(F(\nu), F(\lambda)) = B(F(\nu), F(\lambda))^{G}$

$= (F(\lambda) F(\nu)^*)^{\alpha}$

$\underset{\uparrow}{=} F(e^{\rho}_{11}(\nu, \lambda))$

we show.

∵ $\supset$ OK

$\subset$ $(F(\lambda)F(\nu)^*)^{\alpha} \subset \underset{\text{Claim 3}}{N^{\alpha} = \sigma(M)}$.

$\forall S = \sigma(\overset{\exists}{T}_{\in M}) \in$

Then $\forall a \in (\sigma, \sigma\nu)$

$\sigma(T) a \in (\sigma, \sigma\lambda)$

i.e.

$\sigma(T) a\, \sigma(x) = \sigma\lambda(x)\ \ \sigma(T) a$

$\|$

$\sigma(T)\, \sigma\nu(x)\, a$

Btw. $\mathrm{Irr}\, e^{\rho}_{01} = \{\sigma\}$ & $\sigma\nu = \oplus\sigma$

i.e. $(\sigma, \sigma\nu)$ has Cuntz elem $\sum S_i S_i^* = 1$.

Put $a = S_i$ $\forall i$

$\to$ $\sigma(T\nu(x)) = \sigma(\lambda(x) T)$

$\to$ $T \in (\nu, \lambda)$

Hence. F fully faithful.

On $\otimes$ structure.

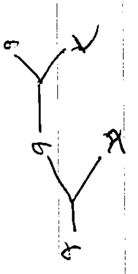
operator prod. in N

$$F(\nu) \underset{\uparrow}{\otimes} F(\lambda) = \left((\sigma, \sigma\nu) \overset{\downarrow}{\cdot} (\sigma, \sigma\lambda) \,,\, \alpha \right)$$

in Rep(G)

$$\underset{(*)}{\subset} \left((\sigma, \sigma\nu\lambda), \alpha \right)$$

$$= F(\nu \otimes \lambda)$$



Btw. $m_\nu := \dim_{\mathbb{C}} (\sigma, \sigma\nu) = d(F(\nu))$

Then $\sigma\nu = m_\nu \bar{\sigma}$ $\quad (\because)\ \mathrm{Irr}\, \mathcal{C}_{01} = \{\sigma\}$.

$\to d(\sigma)\, d(\nu) = m_\nu\, d(\bar{\sigma})$

$\to d(\nu) = m_\nu$ $\quad \forall \nu \in \mathcal{C}_{11}^{\rho}$

Thus. $(*)$ is $=$. by equal dim.

$$F(\nu) \otimes F(\lambda) = F(\nu \otimes \lambda).$$

C^*-tensor functor.

Finally we show the ess. surjectivity of F.

We know

$$\{F(\nu)\}_{\nu \in \mathrm{Irr}\, \mathcal{C}_{11}^{\rho}}$$

are ~~mutually~~ inequivalent. irr. objects. in Rep G

$$\sum_\nu \dim_{\mathbb{C}} F(\nu)^2 = \sum_\nu d(F(\nu))^2$$

$$= \sum_\nu d(\nu)^2$$

$$= \dim \mathcal{C}_{11}^{\rho}$$

$$= \dim \mathcal{C}_{00}^{\rho}$$

$$= |G|.$$

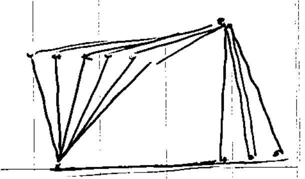
By PW theory, we are done.
(Frob?)

☒

☆ $\mathcal{C}_{00}^{\rho} \cong \mathrm{Hilb}_G$ $\quad \mathcal{C}_{11}^{\rho} \cong \mathrm{Rep}\, G$

Corollary

$\rightsquigarrow \mathrm{Hilb}_G \underset{\text{Morita}}{\cong} \mathrm{Rep}\, G$



§3.2 ADE classification of subfactors.

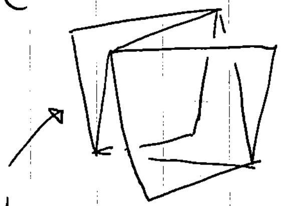
Consider a general situation:

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \mathcal{C}_{10} & \mathcal{C}_{11} \end{pmatrix} \quad \text{rigid } C^*\text{-2-cat.}$$

Take $\rho \in \mathcal{C}_{10}$ & $\mathcal{C}^\rho :=$ the full sub cat of $\mathcal{C}$ gen by ρ.

as before. (cf §3.1) $\mathcal{C}^\rho \subset \begin{pmatrix} \mathrm{Mor} & \mathrm{Mor} \\ \mathrm{Mor} & \mathrm{Mor} \end{pmatrix}$

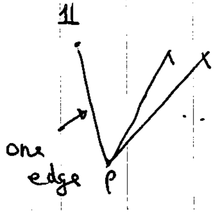
$\rightsquigarrow$ Graph of $\mathcal{C}^\rho$



each adj. graph has norm $\leq d(\rho)$.

Suppose

$d(\rho) < 2$.



$\rightsquigarrow$ $\rho \in \mathrm{Irr}\, \mathcal{C}_{10}^\rho$

(∵) If $\rho = \sigma_1 \oplus \sigma_2$, then

$d(\rho) = d(\sigma_1) + d(\sigma_2) \geq 2$ ↯

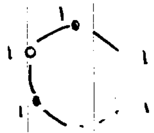
Recall the following:

$\Gamma_2 \subsetneq \Gamma_1$ ← connected bipartite (sub)graphs.

$\rightsquigarrow$ $\|\Lambda_{\Gamma_2}\| \leq \|\Lambda_{\Gamma_1}\|$ (cf. Perron-Frob. theory)

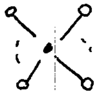
Hence. if a conn. bipartite graph Γ ~~has~~ contain the following graphs, then $\|\Lambda_\Gamma\| \geq 2$.

(1) circuit



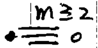
$\begin{bmatrix} & \Lambda \\ \Lambda^t & \end{bmatrix}\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ $\rightsquigarrow$ $\|\Lambda\| = 2$.

(2)



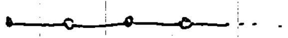
m edges with $m \geq 4$.

$\left\|\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right\| = \sqrt{m} \geq 2$.



$\|(m)\| = m \geq 2$

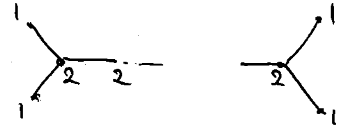
(3) A_∞



Adj mat $= \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{bmatrix} = 1 +$ unilat. shift

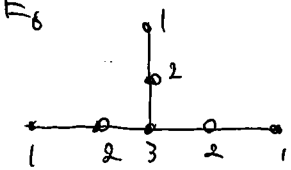
norm $= 2$

(4) Two triple pts



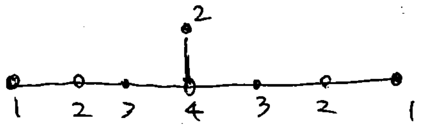
PF eigenvalue = 2
‖
Adj mat.

(5) $E_6^{(1)}$



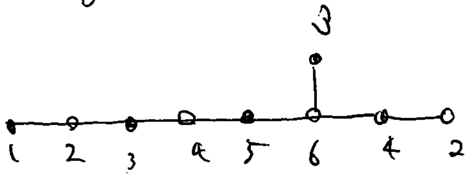
PF eigen = 2

(6) $E_7^{(1)}$



PF eigen = 2

(7) $E_8^{(1)}$



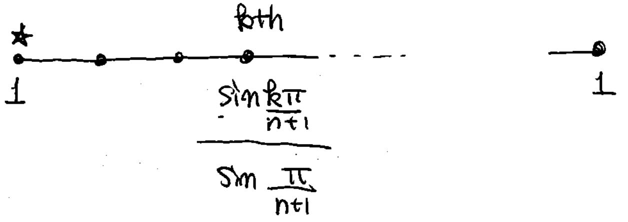
PF eigen = 2

Finite Bipartite Graphs with norm < 2 are

A_n, D_n, E_6, E_7, E_8.
$(n \geq 2)$ $(n \geq 4)$

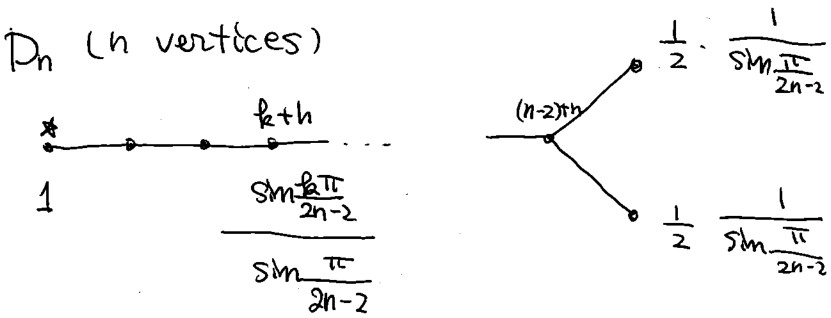
A_n (n vertices)

norm



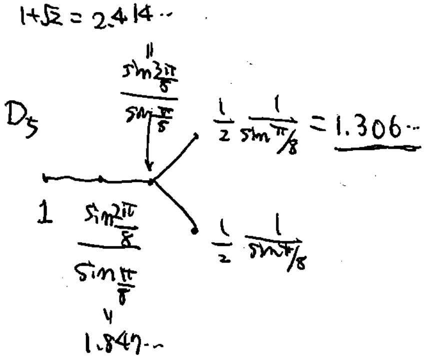
$2\cos\frac{\pi}{n+1}$
$(n \geq 2)$

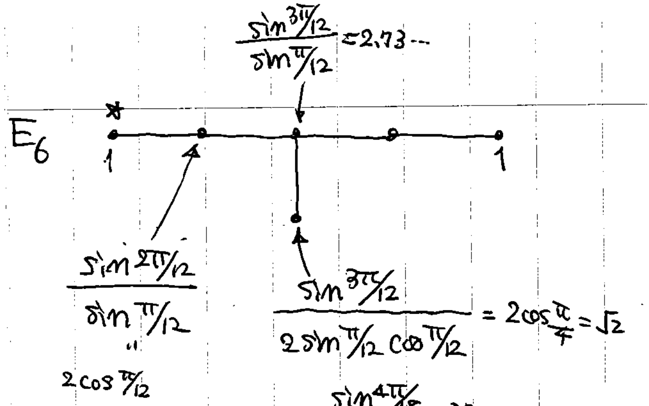
D_n (n vertices)



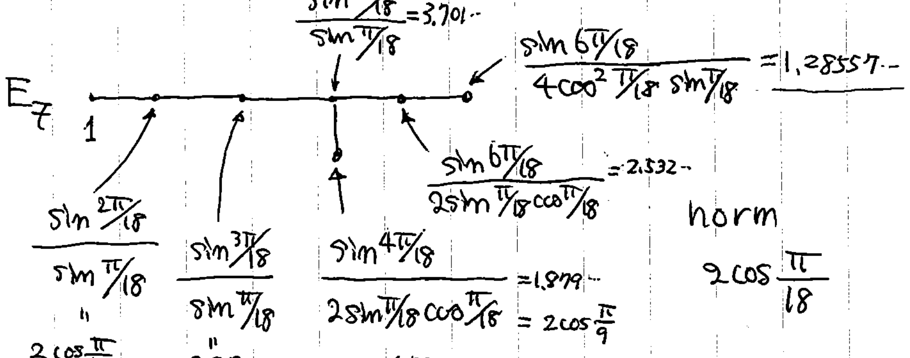
norm

$2\cos\frac{\pi}{2n-2}$
$(n \geq 4)$

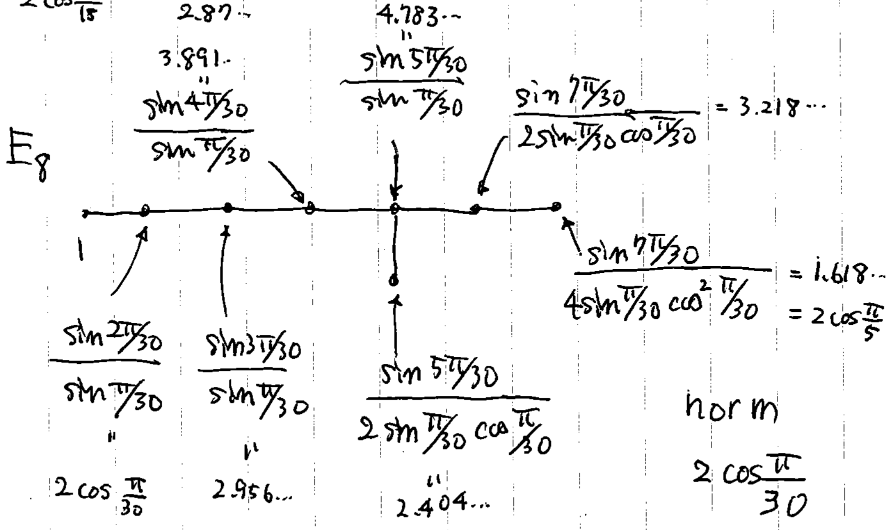




norm $2\cos\frac{\pi}{12}$



norm $2\cos\frac{\pi}{18}$



norm $2\cos\frac{\pi}{30}$

norm $\qquad 2\cos\pi/n \ (n \geqq 3)$

	1	$\sqrt{2}=1.414\cdots$	$1.618\cdots$	$\sqrt{3}$ 1.732	1.801	1.847	$1.879\cdots$	$1.902\cdots$	$1.918\cdots$	$1.931\cdots$ $\overset{"}{}$ $\frac{\sqrt{6}+\sqrt{2}}{2}$	1.9696	$1.989\cdots$
	●	●										
n	3	4	5	6	7	8	9	10	11	12	18	30
	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{17}	A_{29}
				D_4		D_5		D_6		D_7	D_{10}	D_{16}
										E_6	E_7	E_8

Thm. 3.5 (Jones)

$\mathcal{C}$: rigid C^*-(2-) tensor category

$\forall\, X \in \mathcal{C}$

$d(X) \in \{2\cos\pi/n \mid n \geq 3\} \cup [4, \infty)$

Thm. 3.6.

Suppose a connected bipartite finite graph Γ is a graph ass. to a C^*-2-category. Then its PF eigenvector $\vec{d} = (d_k)$ with $d_* = 1$, $d_k \geqq 1$ satisfies

$$d_k \in \{2\cos\frac{\pi}{n}\}_{n\geq 3} \cup [4, \infty)$$

Cor 3.7

$\nexists$ subfactor (or $\overset{\text{rigid}}{C^*\text{-2-cat}}$) of type D_5, E_7. ⌟

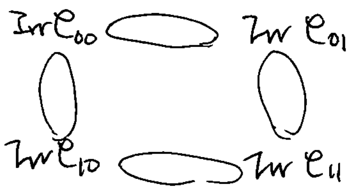
Thm. 3.8

Let $\mathcal{C} = (\mathcal{C}_{rs})_{r,s}$ a rigid C^*-2-cat gen. by $\rho \in \mathcal{C}_{10}$

Suppose $d(\rho) < 2$.

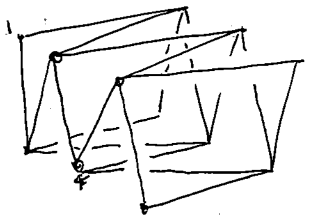
⌞ Then the four graphs are all same ⌟

Proof.



each norm $= d(\rho) < 2$.

If $d(\rho) = 2\cos\frac{\pi}{\text{odd}}$, then A_{2n}



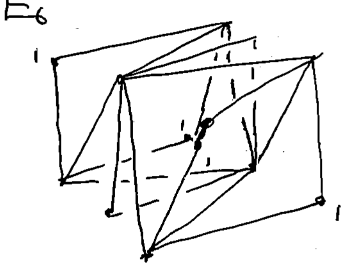
$d(\cdot)$ を見比べると このくっつき方しかない。

他のケースも同様に。

PF eigenvector の値を match させるには

自然なつながり方しかない、ことがわかる。

E_6



Rem 3.9 Known:

$\nexists$ subfactor of D_{odd}. ⌟

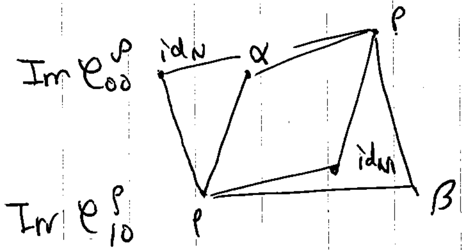
§ A_2, A_3, A_4,

When the graph is A_2 ← norm $2\cos\frac{\pi}{3} = 1$.

$d(\rho) = 1$ i.e. $\rho : N \to M$ $*$-isomo.

Suppose the graph's norm $= 2\cos\frac{\pi}{4} = \sqrt{2}$.

Then the graph is A_3.



Then $d(\rho) = \sqrt{2}$, $d(\alpha) = 1$.

↓

$\alpha : N \to N$ automorph.

$\alpha \in \mathcal{C}^{\rho}_{00} \rightsquigarrow \alpha^2 \in \mathcal{C}^{\rho}_{00}$

$(\alpha \circ \alpha)$

↓

$\alpha^2 \cong id_N$. $\alpha^2 \not\simeq \alpha$

$\alpha \not\simeq id$.

$\bar{\rho}\rho = id_N \oplus \alpha$. $\alpha^2 \cong id_N$

$$\left(\text{cf. } N \xrightarrow{\rho} N \rtimes_{\alpha} \mathbb{Z}/2\mathbb{Z} \quad \bar{\rho}\rho = id_N \oplus \alpha. \right)$$

We want to show

$$N \xrightarrow{\rho} M \cong N \xrightarrow{\pi_\theta} N \rtimes_\theta \mathbb{Z}/2.$$

Generize this situation.

Given a subfactor

$$N \xrightarrow{\rho} M \quad \text{with} \quad \bar{\rho}\rho = \sigma_1 \oplus \cdots \oplus \sigma_n$$

↑ ↗

$\sigma_k \in \mathrm{Aut}(N)$

$\sigma_k \not\cong \sigma_\ell$ if $k \neq \ell$

⇝ $[\sigma] \in \mathrm{Out}(N)$ s.t. $\sigma \prec \bar{\rho}\rho$ forms a finite grp G.

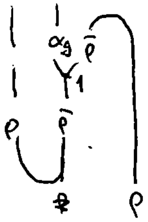
⇝ By cohomology vanishing arg.

WMA $\exists\, G \xrightarrow{\alpha} \mathrm{Aut}(N)$ outer action.

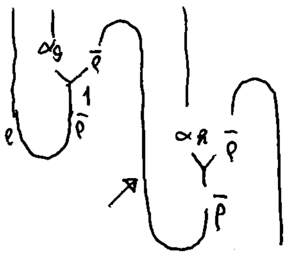
s.t. $\bar{\rho}\rho = \bigoplus_{g \in G} \alpha_g$

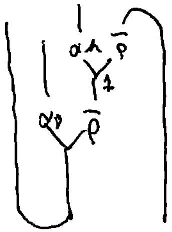
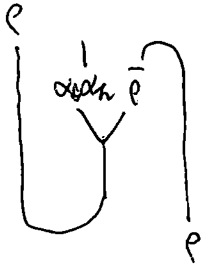
& $\alpha_g \cdot \bar{\rho} = \bar{\rho}$

$U(g) := d(\rho)\, \rho(\alpha_g(R_\rho^*))\, \bar{R}_\rho \in M \qquad g \in G.$

$= d(\rho)$  $\in (\rho, \rho\cdot\alpha_g)$

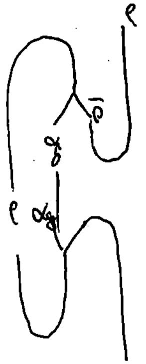
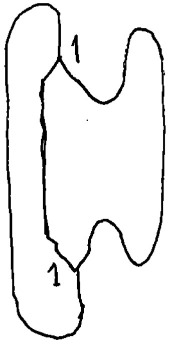
Then $U(g)$ is a unitary rep'n

$U(g)U(h) = d(\rho)^2$ 

$= d(\rho)$  $= d(\rho)$ 

$= U(gh)$

in $(\rho,\rho)=\mathbb{C}$

$U(g)^* U(g) = d(\rho)$  $= d(\rho)$  $= 1.$

Then

$$M = \sum_g \rho(N)U(g)$$

(∵) We know

$$M = \rho(N)\,\bar{R}_\rho$$

Thus enough ~~to~~ show $\bar{R}_\rho \in \sum_g \rho(N)\, u_g$

$\phi_{\bar\rho}\left(\sum_g \alpha_g(R_\rho R_\rho^*)\right)$

$= \bar{R}_\rho\, \rho\left(\sum_g \alpha_g(R_\rho R_\rho^*)\right) \bar{R}_\rho$

$= \sum_g d(\rho)^{-2} = |G|\, d(\rho)^{-2} = 1.$

& $\{\alpha_g(R_\rho R_\rho^*)\}_g$ ortho proj

((∵) $\alpha_g(R_\rho^*)\, \alpha_h(R_\rho) \in (\alpha_h, \alpha_g)$)

$\to \sum_g \alpha_g(R_\rho R_\rho^*) = 1$

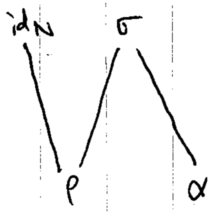
$\to \sum_g \rho(\alpha_g(R_\rho))\, U(g) = d(\rho)\, \bar{R}_\rho$

Thus $N \xrightarrow{\rho} M \cong N \xrightarrow{\pi_\alpha} N \rtimes_\alpha G.$

A4 case

$\mathrm{Irr}\, \mathcal{C}_{00}^{\rho}$

$\mathrm{Irr}\, \mathcal{C}_{10}^{\rho}$



$d(\alpha) = 1 \rightsquigarrow \alpha : N \to M$ $*$-isomo.

$d(\rho) = d(\sigma) = 2\cos\frac{\pi}{5}$

Fusion rule

- $\bar{\rho}\rho = id_N \oplus \sigma$
- $\rho\sigma = \rho \oplus \alpha$
- $\bar{\rho}\alpha = \sigma$

We study the tensor cat $\mathcal{C}_{00}^{\rho}$

$\mathrm{Irr}\, \mathcal{C}_{00}^{\rho} = \{id_N, \sigma\}.$

$$(id_N \oplus \sigma)\sigma = \sigma \oplus \sigma^2$$
$$\|$$
$$\bar{\rho}\rho\sigma = \bar{\rho}\cdot(\rho \oplus \alpha) = \bar{\rho}\rho \oplus \rho\alpha = id_N \oplus 2\sigma$$

$\rightsquigarrow \sigma^2 = id_N \oplus \sigma$

$\sigma \cong \bar{\sigma}$

$[\alpha^{-1}\bar{\rho}] = [\bar{\sigma}] = [\sigma]$

$\to [\bar{\rho}] = [\alpha\bar{\sigma}]$

$\to \rho(N) \subset M \cong \sigma(N) \subseteq N.$

Look at $\sigma^2 = id_N \oplus \sigma$

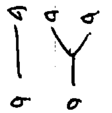
$\exists\, S_1 \in (id_N, \sigma^2)$ $\exists\, S_2 \in (\sigma, \sigma^2)$

s.t.
- $S_1^* S_1 = 1 = S_2^* S_2$
- $S_1 S_1^* + S_2 S_2^* = 1$

Cuntz isometries.

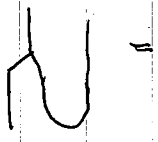
We compute $\sigma(S_1)$ & $\sigma(S_2)$ as follows.

$\sigma(S_1) =$


$\sigma(S_2) =$


$S_1^* \sigma(S_1) =$

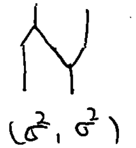
$= c_1 \in \mathbb{C}.$

$S_2^* \sigma(S_2) =$

$= c_2 S_2$

$\in (\sigma, \sigma^2)$

$\to \sigma(S_1) = (S_1S_1^* + S_2S_2^*)\sigma(S_1) = c_1 S_1 + c_2 S_2^2$

$S_1^* \sigma(S_2) =$  $= C_3 S_2^*$

$S_2^* \sigma(S_2) =$  $= C_4 S_1 S_1^* + C_5 S_2 S_2^*.$

$\to \sigma(S_2) = C_3 S_1 S_2^* + C_4 S_2 S_1 S_1^* + C_5 S_2 S_2 S_2^*.$

NEXT, using $\sigma(S_1)^* \sigma(S_1) = 1 = \sigma(S_2)^* \sigma(S_2)$

$\sigma(S_1)\sigma(S_1)^* + \sigma(S_2)\sigma(S_2)^* = 1$

we compute $C_1, \cdots, C_5$.

Then we have

$$\begin{cases} \sigma(S_1) = \frac{1}{d} S_1 + \frac{1}{\sqrt{d}} S_2^2 \\ \sigma(S_2) = \frac{1}{\sqrt{d}} S_1 S_2^* - \frac{1}{d} S_2^2 S_2^* + S_2 S_1 S_1^* \end{cases} \qquad (*)$$

Now forget all things and define $\sigma \in \operatorname{End}(\mathcal{O}_2)$ by $(*)$.

σ commutes with $\gamma_t \in \operatorname{Aut}(\mathcal{O}_2)$

$\gamma_t(S_1) = e^{i2t} S_1, \quad \gamma_t(S_1) = e^{it} S_1$

γ has the unique KMS state $\varphi \in \mathcal{O}_2^*$.

$\to$

σ extends to $M := \pi_\varphi(\mathcal{O}_2)''$

Since

$S_1^* \sigma(S_1) = 1/d.$

$\to \sigma \in \operatorname{End}(M)_0$ & $\sigma = \bar{\sigma}.$

$\underset{\uparrow}{d_M}(\sigma) = ?$

$\operatorname{End}(M)_0$

$\sigma^2 = \mathrm{id} \oplus \sigma$

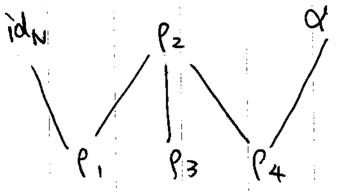
$\to d_M(\sigma)^2 = 1 + d_M(\sigma) \rightsquigarrow d_M(\sigma) = d = 2\cos\frac{\pi}{5}$

Hence $\sigma(M) \subset M$ A_4. type

This is a Cuntz alg. construction due to Izumi.

E_6

$N \xrightarrow{\rho} M.$



$d(\alpha) = 1,$

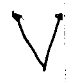
$d(\rho_1) = d(\rho_4) = 2\cos\frac{\pi}{12} < 2.$

$d(\rho_2) = \dfrac{\sin\frac{3}{12}\pi}{\sin\frac{\pi}{12}} \sim 3$

$d(\rho_3) = 2\cos\frac{\pi}{4}. < 2$

$\uparrow$

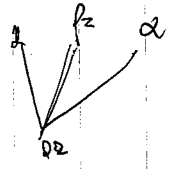
A_3



$\mathbb{Z}_2$ action.

$\rho_3(N) \subset M. \cong \rho_3(N) \subset \rho_3(N) \rtimes \mathbb{Z}_2$

~~$d(\rho_2)$~~



$\bar{\rho}_3\rho_3 = id \oplus \alpha.$

$\bar{\rho}_1\rho_1 = id \oplus \rho_2$

$\bar{\rho}_1\rho_3 = \rho_2$
$\quad\;\, ε \qquad \|$
$\qquad\;\; \bar{\rho}_3\rho_1$

$\alpha^2 = 1.$

$\rho_2^2 = 1 + 2\rho_2 + \alpha$

$\alpha\rho_2 = \rho_2\alpha = \rho_2$

$\rho_1\alpha = \rho_4$

$\rho_1\rho_2 = \rho_1 + \rho_3 + \rho_4$

$$\begin{aligned}
&\bar{\rho}_1\rho_1\rho_2 = \bar{\rho}_1\rho_1 + \bar{\rho}_1\rho_3 + \bar{\rho}_1\rho_4 \\
&\quad\| \qquad\qquad\qquad\qquad \| \\
&(1 \oplus \rho_2)\rho_2 \qquad 1 \oplus \rho_2 \oplus \rho_2 \oplus \rho_2 + \alpha \\
&\quad\| \\
&\rho_2 \oplus \rho_2^2
\end{aligned}$$

$\to \rho_2^2 = 1 \oplus 2\rho_2 \oplus \alpha$

$\bar{\rho}_1\rho_4 = \rho_2 \oplus \alpha$

$(\alpha, \bar{\rho}_1\rho_4) \cong (\rho_1\alpha, \rho_4)$

$$\bar{\rho}_1\rho_3(N) \overset{\alpha\,\mathbb{Z}_2}{\curvearrowright}\subset \bar{\rho}_1(M) \subset N$$
$$\|$$
$$\rho_2(N)$$

$(1, \rho_2, \alpha)$ を実現. O_4 で, それから crossed. prod. を取る.

$\mathcal{O}_4$ 上で ρ_2, α ができたとする。

Extend $\rightarrow$ M 上の ρ_2^M, α^M にのばす。

$$d(\rho_2^M) \leqq d(\rho_2) = \frac{\sin 3\pi/12}{\sin \pi/12}$$

↖今回は2以上なので ρ_2^M がキヤクであることはここからはいえない。

まず $\mathbb{1}$, α, ρ_2 で生成されるテンソル圏 $\mathcal{C}$ を $\mathrm{End}(\mathcal{O}_4)_0$ で考える。このほで fusion rule がちゃんと同じになっているかを考える。

これができれば、明らかに amenable category だから、次元の最小性により、$d(\rho_2^M) = d(\rho_2)$ が従う。

$$\mathcal{C}^{\rho_2} \longrightarrow \mathrm{End}(M)_0$$

$$\rho_2^2 = \mathbb{1} + \alpha + \rho_2$$

$$\downarrow$$

$\rho_2^{M2} = \mathbb{1} + \alpha^M + \rho_2^M$ $\dim(\rho_2^{M2}, \mathbb{1}) \geqq 2$ より。

ρ_2^M キヤクでなければ、$\rho_2^M > \mathbb{1}$ でなくてはならない。

$$\rho_2^M = \cancel{\mathbb{1}+}\, n\mathbb{1} + \sigma + \cdots$$

$$v \in (\mathbb{1}_M, \rho_2^M) \subset M.$$

$$vx = \rho_2^M(x)v \quad \forall x \in \mathcal{O}_4$$

$$\rho_2^M = k\mathbb{1} + \sigma$$

↑のこりまとめたもの

$$\rho_2^{M2} = k^2\mathbb{1} + 2k\sigma + \sigma^2 = \mathbb{1} + \alpha + \rho_2^M = \mathbb{1} + \alpha + k\mathbb{1} + \sigma = (k+1)\mathbb{1} + \alpha + \sigma$$

$$\rightsquigarrow \quad \sigma^2 = (-k^2 + \overset{k}{\cancel{2k}} + 1)\mathbb{1} \cancel{-k\sigma} + (1-2k)\sigma + \alpha$$

$\rightsquigarrow$ $k = 1$ しかありえない

$$\rho_2 = \mathbb{1} + \sigma. \quad \& \quad \sigma^2 = \mathbb{1} + \alpha.$$

$\rightarrow \; d(\sigma)^2 = 2 \qquad d(\sigma) = \sqrt{2}.$

しかし $d(\rho_2) \neq 1+\sqrt{2}.$

ちなみに.

$\sin 3\pi/12 = \sin\left(\frac{2\pi}{12} + \frac{\pi}{12}\right)$

$= \sin$

$\sin 3\theta = \frac{e^{3i\theta} - e^{-3i\theta}}{2i}$

$= \frac{(e^{i\theta} - e^{-i\theta})^3}{2i} + \frac{3e^{i\theta} - 3e^{-i\theta}}{2i}$

$= 3\sin\theta - 4\sin^3\theta.$

$\frac{\sin 3\pi/12}{\sin \pi/12} = \frac{3\sin\pi/12 \; - 4\sin^3\pi/12}{—}$

$= 3 - 4\sin^2\pi/12$

$= 3 - 4 \cdot \frac{1-\cos\frac{\pi}{6}}{2}$

$= 3 - 2\left(1 - \frac{\sqrt{3}}{2}\right)$

$= 1 + \sqrt{3}$

したがって. ρ_2^M は キヤクである.

このことから $\mathcal{C}^{\rho_2} \longrightarrow \mathrm{End}(M)_0$

は fully faithful であることが分かり, subfactor の

実現ができていることになる.

$\sigma^2 = (-k^2+k+1)\mathbb{1} + \alpha + (1-2k]\sigma$

<u>case</u> $\alpha = \mathbb{1}.$

$\sigma^2 = (-k^2+k+2)\mathbb{1} + (1-2k)\sigma \qquad \rightarrow \alpha \neq \mathbb{1}$

$\uparrow \sigma \neq \mathbb{1}$ より

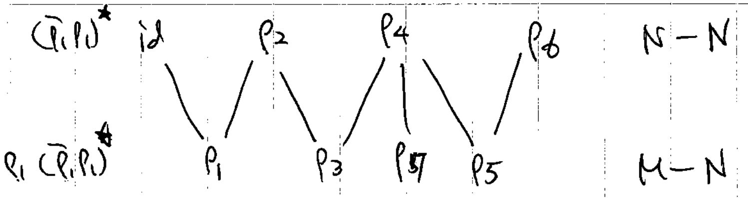
ここがマイナスになり $\oint$

<u>case</u> $\alpha = \sigma$

$\sigma^2 = (-k^2+k+1)\mathbb{1} + (2-2k)\alpha$

$\rightarrow k = 1, \; \sigma^2 = \mathbb{1}.$

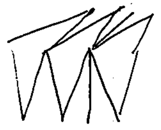
E_8



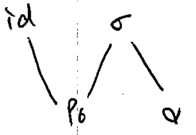
$d(\rho_1) = 2\cos\frac{\pi}{30} \quad < 2.$

$d(\rho_6) = 2\cos\frac{\pi}{5} \quad < 2$

Coxeter nb $= 5$ → ρ_6 A_4



(ρ_6 のなす部因子 →



)

$$\begin{tikzcd} {} \arrow[r, "\alpha\bar\rho_1"] \arrow[d, "\rho_1\otimes"'] & \bar\rho_3 \arrow[d, "\rho_1\bar\rho_3"] \\ \rho_3 \arrow[r, "\alpha\bar\rho_1"'] & {} \end{tikzcd}$$

$\bar\rho_1\rho_1 = 1 + \rho_2$ → $\bar\rho_2 = \rho_2$

$\bar\rho_4 = \rho_4$

$\bar\rho_6 = \rho_6$

$\rho_1\rho_2 = \rho_1 + \rho_3$

$\bar\rho_1\rho_3 = \rho_2 + \rho_4$

$\rho_1\rho_4 = \rho_3 + \rho_7 + \rho_5$

$\bar\rho_1\rho_5 = \rho_4 + \rho_6$

$\rho_1\rho_6 = \rho_5$

$$M \supset \rho_1(N) \supset \rho_1\rho_6(N) \xleftarrow{} A_4$$

$$\rho_1\rho_6(N) = \rho_5(N)$$

intermediate subfactor.

ρ_5 のなす圏をしらべる必要あり.

$\bar\rho_5\rho_5 = \bar\rho_6\bar\rho_1\rho_1\rho_6 = \bar\rho_6\rho_6 + \bar\rho_6\rho_2\rho_6$

$=$

module category の構造をしらべる.

$(\rho_3\rho_2, \rho_1) \cong (\rho_2, \bar\rho_3\rho_1) \cong (\rho_2, \bar\rho_1\rho_3)$ *one dim

$(\rho_3\rho_2, \rho_k) \cong (\rho_2, \bar\rho_k\rho_3)$

$\rho_3\rho_2 = \rho_1 + n_k\rho_k + \cdots$

$\bar\rho_1\rho_3\rho_2 = \bar\rho_1\rho_1 + n_k\bar\rho_1\rho_k + \cdots$

$\bar\rho_1\rho_1 = 1 + \rho_2$

$\bar\rho_1\rho_3\rho_2 = (\rho_2 + \rho_4)\rho_2 = \rho_2^2 + \rho_4\rho_2$

$(\rho_7\rho_6, \rho_1) \cong (\bar\rho_1\rho_7, \rho_6) = 0.$

$\bar\rho_1(\rho_7\rho_6, \rho_3) \subset (\bar\rho_1\rho_7\rho_6, \bar\rho_1\rho_3)$

$(\bar\rho_1\rho_7\rho_6, \bar\rho_1\rho_3) = (\rho_4\rho_6, \rho_2 + \rho_4)$

$\bar\rho_1(\rho_7\rho_6, \rho_7) \subset (\rho_4\rho_6, \rho_4\rho_6)$

$\bar\rho_7$

$(\rho_3\rho_2, \rho_3) = ?$

$\rho_3\bar\rho_1\rho_1 = \rho_3 + \rho_3\rho_2$

$\rho_3\bar\rho_1\rho_1 = \rho_1\bar\rho_3\rho_1 = (\bar\rho_2 + \bar\rho_4)\rho_1$

$(\bar\rho_2 + \bar\rho_4)\rho_1 = \rho_1 + 2\rho_3 + \rho_7 + \rho_5$

$\rho_3\rho_2 = \rho_1 + \rho_3 + \rho_7 + \rho_5$

$\rho_3\bar\rho_1\rho_3 = \rho_3\rho_2 + \rho_3\rho_4$

$=$

(

$\rho^2 = \sum_{g\in G} \alpha_g + n\rho$　　$\rho \in \mathrm{End}(\mathcal{O}_{|G|+n})$

$\rho = \sum m_k \sigma_k$ on M.　$G \curvearrowright (\rho, \rho^2)$

$\rho^2 = \underbrace{\sum m_k^2 \mathbb{1}} + \cdots$

↑ ρがきやくでなければ 2以上.

$= \sum \alpha_g + n\rho$

↑ 1は1つ

$\to \rho = m_1 \mathbb{1} + \sum_{k\geq 2} m_k \sigma_k$ であり.

$\sum_{k\geq 1} m_k^2 = 1 + n m_1$

また $\alpha_g \rho = \rho$ on $\mathcal{O}_n$

だから　$\rho > \alpha_g$ である.

$\rho = \sum_{g\in e} m_g \alpha_g + \sum_k m_k \sigma_k$　↙ α_gでないもの達.

とかく　$(\alpha_g, \rho) \cong (\mathbb{1}, \alpha_g \rho) = (\mathbb{1}, \rho)$

より

$\rho = m \sum_{g\in G} \alpha_g + \sum_k m_k \sigma_k$

$\rho^2 = m^2 |G| \sum_g \alpha_g + 2m|G| \sum_k m_k \sigma_k$

$+ \sum_{k,l} m_k m_l \underset{\sim}{\sigma_k \sigma_l}$

↓ ここがα_gをふくむかもしれない

$= ~~m\sum~~$

$(1+nm) \sum_g \alpha_g + n \sum_k m_k \sigma_k$

For simplicity $n \leq |G|$

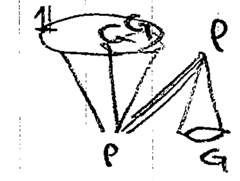
$m^2|G| \leq 1+nm \rightsquigarrow m=1$ しかありえない.

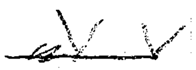
$\rho^2 = |G| \sum_g \alpha_g + \textcircled{2|G|} \sum m_k \sigma_k + \sum m_k m_l \sigma_k \sigma_l$

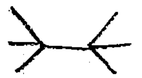
$= (1+n) \sum_g \alpha_g + \textcircled{n} \sum m_k \sigma_k$

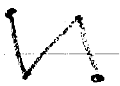
$\to 2|G| \leq n|G|$　これはありえない.

したがって ρ はきやくでなくてはならない









Section 4 C^*-2-categories & Std. lattices

$$\begin{tikzcd}
 & \text{Subfactors} \arrow[d] & \\
 & \text{singly generated rigid } C^*\text{-2-cat} \arrow[dl, shift left, "①"'] \arrow[dr] & \\
\text{std. lattice} \arrow[ur, shift left] \arrow[rr, shift left] & & \text{Paragrp} \arrow[ll, shift left]
\end{tikzcd}$$

§4.1 C^*-2-cat to std lattice ①

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \mathcal{C}_{10} & \mathcal{C}_{11} \end{pmatrix}$$

$\rho \in \mathcal{C}_{10}$ a generator of $\mathcal{C}$.

We will simply write $\sigma\mu$ for $\sigma \otimes \mu$.

Fix a conj obj: $\bar{\rho} \in \mathcal{C}_{01}$

& a $\underset{\text{std}}{\text{solution}}$ of conj. eq. $(R_\rho, \bar{R}_\rho)$

$R_\rho \in \mathcal{C}_{00}(\mathbb{1}, \bar{\rho}\rho)$ ← isometries

$\bar{R}_\rho^* \in \mathcal{C}_{11}(\mathbb{1}, \rho\bar{\rho})$

$(R_\rho^* \otimes 1_{\bar{\rho}})(1_{\bar{\rho}} \otimes \bar{R}_\rho) = 1/d(\rho)$

$(\bar{R}_\rho^* \otimes 1_\rho)(1_\rho \otimes R_\rho) = 1/d(\rho)$

Now we set C^*-algs

- $A_{0,0} := \mathcal{C}_{00}(\mathbb{1}, \mathbb{1})$, $A_{0,1} := \mathcal{C}_{01}(\bar{\rho}, \bar{\rho})$, $A_{0,2} := \mathcal{C}_{00}(\bar{\rho}\rho, \bar{\rho}\rho)$, …

$\rightsquigarrow A_{0,2n} := \mathcal{C}_{00}((\bar{\rho}\rho)^n, (\bar{\rho}\rho)^n)$, $A_{0,2n+1} := \mathcal{C}_{01}((\bar{\rho}\rho)^n\bar{\rho}, (\bar{\rho}\rho)^n\bar{\rho})$

- $A_{1,-1} := \mathcal{C}_{11}(\mathbb{1}, \mathbb{1})$, $A_{1,0} := \mathcal{C}_{10}(\rho, \rho)$, $A_{1,1} := \mathcal{C}_{11}((\rho\bar{\rho}), (\rho\bar{\rho}))$, …

$\rightsquigarrow A_{1,2n-1} := \mathcal{C}_{11}((\rho\bar{\rho})^n, (\rho\bar{\rho})^n)$, $A_{1,2n} := \mathcal{C}_{10}((\rho\bar{\rho})^n\rho, (\rho\bar{\rho})^n\rho)$.

Then we get

$$\begin{tikzcd}
 & A_{0,0} \arrow[r, "\cdot\otimes 1_{\bar{\rho}}"] \arrow[d, "1_\rho\otimes\cdot"'] & A_{0,1} \arrow[r, "\cdot\otimes 1_\rho"] \arrow[d, "1_\rho\otimes\cdot"'] & A_{0,2} \arrow[r, "\cdot\otimes 1_{\bar{\rho}}"] \arrow[d, "1_\rho\otimes\cdot"'] & \cdots \\
A_{1,-1} \arrow[r, "\cdot\otimes 1_\rho"'] & A_{1,0} \arrow[r, "\cdot\otimes 1_{\bar{\rho}}"'] & A_{1,1} \arrow[r, "\cdot\otimes 1_\rho"'] & A_{1,2} \arrow[r, "\cdot\otimes 1_{\bar{\rho}}"'] & \cdots
\end{tikzcd}$$

The diagrams are commuting.

This nest of C^*-algs satisfies the axiom of <u>lattice</u> due to Popa.

Defn. 4.1

Let $0 < \lambda \leqq 1$.

Let

$$\begin{tikzcd}
 & A_{00} \arrow[r, "i_0"] \arrow[d, "k_0"'] & A_{01} \arrow[r, "i_1"] \arrow[d, "k_1"'] & A_{02} \arrow[r] \arrow[d, "k_2"'] & \cdots \\
A_{1-1} \arrow[r, "j_{-1}"'] & A_{10} \arrow[r, "j_0"'] & A_{11} \arrow[r, "j_1"'] & A_{12} \arrow[r] & \cdots
\end{tikzcd}$$

be a nest of fin dim C^*-algs such that

(1) $k_{n+1} \circ i_n = j_n \circ k_n$ (comm. diagram)

(2) Each i_n, j_n, k_n has left inverses

$\phi_{i_n} \cdot \phi_{j_n}, \phi_{k_n}$ s.t.

$$\phi_{j_{n-1}} \circ k_n = k_{n-1} \circ \phi_{i_{n-1}}$$

(comm. square).

(3) Jones projections: e_n $(n \geqq 2)$.
f_m $(m \geqq 1)$.

- $f_1 \in A_{11}$
- $e_n \in A_{0n}$ $(n \geqq 2)$
- $k_n(e_n) = f_n$

s.t.

- $f_{n+1} j_n(y) f_{n+1} = j_{n-1}(\phi_{j_{n-1}}(y)) f_{n+1}$ $\forall y \in A_{1n}$

($\overset{\text{implies}}{\rightsquigarrow}$ $e_{n+1} i_n(x) e_{n+1} = i_{n-1}(\phi_{i_{n-1}}(x)) e_{n+1}$ $\forall x \in A_{in}$)

- Markov property (Push-Down technique)

– Horizontal

$$\frac{1}{\lambda} f_n E_{j_{n-1}}(f_n x) = f_n x$$

$$\begin{tikzcd}
A_{1n-1} \arrow[r, "j_{n-1}"'] & A_{1n} \arrow[d, phantom, "\ni" rotate=-90] \\
 & \forall x
\end{tikzcd}$$

$\forall n \geqq 1$ $\quad f_n$

($\rightsquigarrow$ $\frac{1}{\lambda} e_n E_{i_{n-1}}(e_n x) = e_n x$ & $E_{j_{n-1}}(f_n) = \lambda$

– Vertical.

$$\frac{1}{\lambda} f_1 E_{k_n}(f_1 x) = f_1 x \quad \forall x \in A_{1n}$$

$\uparrow$

More precisely,
$j_{n-1} \circ \cdots \circ j_1(f_1)$

$$\begin{tikzcd}
 & A_{0n} \arrow[d, "k_n"] \\
A_{11} \arrow[r] \arrow[d, phantom, "\ni" rotate=-90] & \cdots \arrow[r] & A_{1n} \arrow[d, phantom, "\ni" rotate=-90] \\
f_1 & & x
\end{tikzcd}$$

$E_{k_1}(f_1) = \lambda$.

$\rightarrow$ e_n & f_n satisfies the Temperly–Lieb rel.

(4) $k_n(\varphi_0) = \varphi_1$

The system $\{A_{ij}\}_{ij}$ with these properties is called the <u>λ-sequence</u> ~~of~~ or <u>λ-lattice</u>

1

For $\mathcal{C} = (\mathcal{C}_{rs})_{r,s}$ generated by $\rho \in \mathcal{C}_{10}$,

the associated system

$$\begin{tikzcd}
 & \mathcal{C}_{00}(1\!\!1, 1\!\!1) \arrow[r, "i_0"] \arrow[d, "\rho"'] & \mathcal{C}_{01}(\bar{\rho}, \bar{\rho}) \arrow[r, "i_1"] \arrow[d, "\rho"'] & \mathcal{C}_{00}(\bar{\rho}\rho, \bar{\rho}\rho) \arrow[r] \arrow[d, "\rho"'] & \cdots \\
\mathcal{C}_{11}(1\!\!1, 1\!\!1) \arrow[r] & \mathcal{C}_{10}(\rho, \rho) \arrow[r, "j_0"'] & \mathcal{C}_{11}(\rho\bar{\rho}, \rho\bar{\rho}) \arrow[r, "j_1"'] & \mathcal{C}_{10}(\rho\bar{\rho}\rho, \rho\bar{\rho}\rho) \arrow[r] & \cdots
\end{tikzcd}$$

satisfies the axiom of λ-lattice.

we check!

- The left inverses are defined by

$$\phi_{i_{2n}}(x) = (1_{\bar{\rho}\rho\cdots\bar{\rho}} \otimes R_\rho^*)(x \otimes 1_\rho)(1 \otimes R_\rho)$$

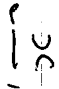
$$\left(= (\bar{\rho}\rho)^n (R_\rho^*) \; x \; (\bar{\rho}\rho)^n (R_\rho) \right)$$

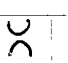
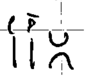
etc.

$=$  $x \in \mathcal{C}_{01}((\bar{\rho}\rho)^n\bar{\rho}, (\bar{\rho}\rho)^n\bar{\rho})$

- Jones projs are

$f_1 = \bar{R}_\rho \bar{R}_\rho^* =$ 

$e_2 = R_\rho R_\rho^* =$  $f_2 =$ 

$e_3 = 1 \bullet R_\rho R_\rho^* =$  $\cdots$ $f_3 =$ 

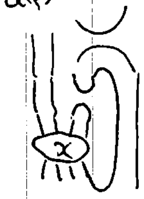
- $f_{n+1} j_{n\cancel{+1}}(y) f_{n+1} = j_{n-1} \phi_{j_{n-1}}(y) f_{n+1}$ is also ok

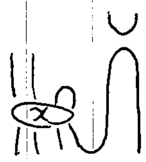
(∵) $f_3 j_2(y) f_3 =$ 

$= j_1(\phi_{j_1}(y)) f_3$.

- Markov prop.

– Horizontal. with $\lambda = d(\rho)^{-2}$

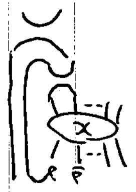
(∵) $\frac{1}{\lambda} f_3 E_{j_2}(f_3 x) = \frac{1}{\lambda}$ 

$= \frac{1}{\lambda} \cdot \frac{1}{d(\rho)}$ 

$= \frac{1}{\lambda d(\rho)^2}$ 

$= f_3 x$.

– vertical.

(∵) $\frac{1}{\lambda} f_1 E_{k_n}(f_1 x) = \frac{1}{\lambda}$ 

$= \frac{1}{\lambda d(\rho)}$ 

$= \frac{1}{\lambda d(\rho)^2}$ 

$= f_1 x$.

目標

C^*-2-category with a generator ρ.

を構成する ⌟

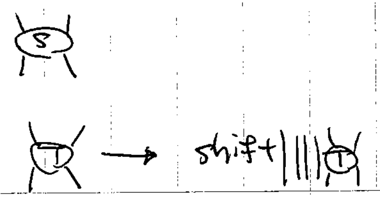
$A_0 := \varinjlim A_{0n} \supset \cdots \supset A_{0n} \supset A_{0n-1} \supset \cdots$

$A_1 := \varinjlim A_{1n} \supset \cdots \supset A_{1n} \supset A_{1n-1} \supset \cdots$

とみなす.

$$\begin{tikzcd} A_0 \arrow[d, "\rho"] \\ A_1 \end{tikzcd}$$

$S \in A_{1n}$ と $T \in A_{1m}$ のテンソル積 $S \otimes T$ を定ぎしたい.



§4.2. Corner endomorphisms (canonical shift)

Set

$v_n := \frac{1}{\sqrt{\lambda}^{n-2}} e_2 \cdots e_n \in A_{0n}$ $\quad n \geq 2$

$w_n := \frac{1}{\sqrt{\lambda}^{n-1}} f_1 \cdots f_n \in A_{1n}$ $\quad n \geq 1$

Lem. 4.2

(1) $v_n^* v_n = e_n$, $v_n v_n^* = e_2$

(2) $w_n^* w_n = f_n$, $w_n w_n^* = f_1$. ⌟

Lem. 4.3

(1) $\forall x \in A_{0n}$. $\forall n \geq 0$.

$v_m x v_m^* = v_{n+2} x v_{n+2}^*$ $\quad \forall m \geq n+2$.

(2) $\forall y \in A_{1n}$ $\forall n \geq 0$

$w_m y w_m^* = w_{n+2} y w_{n+2}^*$ ⌟

(1) $v_{n+3}\, x\, v_{n+3}^{*}$

$$= \frac{1}{\lambda^{n+1}} e_2 \cdots \underline{e_{n+2}e_{n+3}}\, x\, \underbrace{e_{n+3}e_{n+2}}_{\to\ \lambda e_{n+2}} \cdots e_2$$

$$= \frac{1}{\lambda^{n+1-1}} e_2 \cdots e_{n+2}\, x\, e_{n+2} \cdots e_2$$

$$= v_{n+2}\, x\, v_{n+2}^{*}$$

(2) $w_{n+3}\, y\, w_{n+3}^{*}$

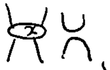
$$= \frac{1}{\lambda^{n+2}} f_1 \cdots \underline{f_{n+2}f_{n+3}}\, y\, \underline{f_{n+3}f_{n+2}} \cdots f_1$$

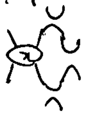
$$= \frac{1}{\lambda^{n+1}} f_1 \cdots f_{n+2}\, y\, f_{n+2} \cdots f_1$$

$$= w_{n+2}\, y\, w_{n+2}^{*}$$

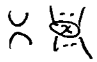
☆

given

$e_{n+2}\, x\, e_{n+2} =$ 

$e_{n+1}e_{n+2}\, x\, e_{n+2}e_{n+1} =$  $= d(\bar{p})^2$ 

⋮

$e_2 \cdots e_{n+2}\, x\, e_{n+2} \cdots e_2 = d(p)^{*}$ 

Defn. 4.4

- $\Phi_0 : A_0 \to A_0$ $*$-homo.

$$\Phi_0(x) = \lim_{n\to\infty} v_n\, x\, v_n^{*} \qquad x \in A_0$$

$$\Phi_0(1) = e_2$$

- $\Phi_1 : A_1 \to A_1$ $*$-homo

$$\Phi_1(x) = \lim_{n\to\infty} w_n\, x\, w_n^{*} \qquad x \in A_1$$

$$\Phi_1(1) = f_1$$

Lem.

(1) $\Phi_0(A_0) = e_2 A_0 e_2$

(2) $\Phi_1(A_1) = f_1 A_1 f_1$

⌋

Proof.

(1) $\subset$ Trivial.

$\supset$ $x \in A_0$ on

~~$\Phi_0(v_{n+2}^{*}\, x\, v_{n+2})$~~

Put $z := E_n(v_{n+1}^* x v_{n+1}) \frac{1}{\lambda}$.

$\in A_{0,n}$

$$\begin{tikzcd} A_{0n} \arrow[r] & A_{0n+1} \arrow[r] \arrow[l, bend right, "E_n"'] & A_{0n+2} \\ & & e_{n+2} \arrow[u, "\in"] \end{tikzcd}$$

$$\Phi_0(z) = v_{n+2}\, z\, v_{n+2}^*$$

$$= \frac{1}{\lambda} \underset{\overset{\uparrow}{e_{n+2}}}{v_{n+2}} E_n(v_{n+1}^* x v_{n+1}) \underset{\overset{\uparrow}{e_{n+2}}}{v_{n+2}^*}$$

$$= \frac{1}{\lambda} v_{n+2}\, v_{n+1}^*\, x\, v_{n+1}\, v_{n+2}^*$$

$$\overset{\uparrow}{e_{n+1}} e_{n+2} e_{n+1}$$

$$= \frac{1}{\lambda} \frac{1}{\sqrt{\lambda}^2} v_{n+1} \underset{\lambda}{\uparrow} v_{n+1}^* \; x \; v_{n+1}^* \underset{\lambda}{\uparrow} v_{n+1}$$

$$= e_2 x e_2$$

1 (2) $\subset$ Trivial

$\supset$ $y \in A_{1,n}$.

Put $z := \frac{1}{\lambda} E_n(w_{n+1}^* x w_{n+1})$.

$$\begin{tikzcd} A_{1n} \arrow[r] & A_{1n+1} \arrow[r] & A_{1n+2} \\ x \arrow[u, "\in"] & & f_{n+2} \arrow[u, "\in"] \end{tikzcd}$$

$$\Phi_1(z) = w_{n+2}\, z\, w_{n+2}^*$$

$$= \frac{1}{\lambda} w_{n+2}\, w_{n+1}^*\, x\, w_{n+1}\, w_{n+2}^*$$

$$f_{n+1} f_{n+2} f_{n+1}$$

$$= \frac{1}{\lambda} \cdot \frac{1}{\sqrt{\lambda}^2} w_{n+1} \lambda w_{n+1}^* \; x \; w_{n+1} \lambda w_{n+1}^*$$

$$= f_1 x f_1$$

☆ $e_2 \neq 1 \neq f_1$ since $\lambda \lneq 1$

$\rightsquigarrow \Phi_0, \Phi_1$ corner endos.

☆ $\Phi_0(e_n) = e_2 e_{n+2}$

$\Phi_1(f_n) = f_1 f_{n+2}$

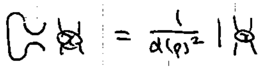
§4.3 shift embeddings

We will construct.

$$\begin{tikzcd}
 & & \mathcal{A}_{0,0} \arrow[r] \arrow[d, "k_0"'] & \mathcal{A}_{0,1} \arrow[r] \arrow[d] & \mathcal{A}_{0,2} \arrow[r] \arrow[d] & \cdots \\
 & \mathcal{A}_{1,-1} \arrow[r] \arrow[d, "l_{-1}"'] & \mathcal{A}_{1,0} \arrow[r] \arrow[d, "l_0"'] & \mathcal{A}_{1,1} \arrow[r] \arrow[d, "l_1"'] & \mathcal{A}_{1,2} \arrow[r] \arrow[d, "l_2"'] & \cdots \\
\mathcal{A}_{0,0} \arrow[r] & \mathcal{A}_{0,1} \arrow[r] & \mathcal{A}_{0,2} \arrow[r] & \mathcal{A}_{0,3} \arrow[r] & \mathcal{A}_{0,4} \arrow[r] & \cdots
\end{tikzcd}$$

$$l_n(x) := \lambda^{-1} \phi_{k_{n+2}} (\Phi_1(x)) \quad (n \geq -1)$$

$$\begin{tikzcd}
 & & \mathcal{A}_{0\,n+2} \\
\mathcal{A}_{1,n} \arrow[rru, "l_n"] \arrow[rr, "\Phi_1"] & & \mathcal{A}_{1\,n+2} \arrow[u, "\phi_{k_{n+2}}"'] \\
x \arrow[u, phantom, "\in" rotate=90] & &
\end{tikzcd}$$



Lem. 4.5

(1) $l_n : \mathcal{A}_{1n} \to \mathcal{A}_{0\,n+2}$ unital $*$-homo. faithful

(2) $l_n(f_n) = e_{n+2}$.

Proof.

(1) $x, y \in \mathcal{A}_{1n}$.

$$l_n(x)\, l_n(y) = \lambda^{-2} \phi_{k_{n+2}} \left(\Phi_1(x)\, k_{n+2} \phi_{k_{n+2}}(\Phi_1(y)) \right)$$

$$\Phi_1(x)\, E_{k_{n+2}}(\Phi_1(y))$$

$$= \Phi_1(x)\, f_1\, E_{k_{n+2}}(f_1 \Phi_1(y))$$

$$= \Phi_1(x)\, \lambda\, f_1\, \Phi_1(y) \qquad \text{(vertical Markov)}$$

$$= \lambda \Phi_1(xy)$$

$$\rightsquigarrow l_n(x)\, l_n(y) = \lambda^{-2} \phi_{k_{n+2}}(\lambda \Phi_1(xy))$$

$$= l_n(xy).$$

$$l_n(1) = \lambda^{-1} \phi_{k_{n+2}}(f_1) = 1$$

$$l_n(x^*x) = 0 \iff \Phi_1(x) = 0 \iff x W_{n+2}^* = 0$$

$$\iff \underset{\overset{\uparrow}{\mathcal{A}_{1,n+2}}}{x} f_{n+2} = 0 \implies x = 0.$$

(2) $$l_n(f_n) = \lambda^{-1}\phi_{k_{n+2}}(\Psi_1(f_n))$$

$$= \lambda^{-1}\phi_{k_{n+2}}(w_{n+2} f_n w_{n+2}^*)$$

$$\Psi_1(f_n) = w_{n+2} f_n w_{n+2}^*$$

$$= \frac{1}{\lambda^{n+1}} f_1 \cdots f_n f_{n+1} f_{n+2} f_n \cdot f_{n+2} f_{n+1} f_n \cdots f_1$$

$$= \frac{1}{\lambda^{n}} f_1 \cdots f_n f_{n+2} f_{n+1} f_n \cdots f_1$$

$$= \frac{1}{\lambda^{n\cancel{+1}}} f_1 \cdots f_n \boxed{f_{n+3}} f_n \cdots f_1$$

$$= \cancel{\frac{1}{\lambda^2}} w_n w_n^* f_{n+2}$$

$$= f_1 f_{n+2} \qquad k_{n+2}(e_{n+2})$$

$$\qquad\qquad\qquad k \quad \|$$

$$l_n(f_n) = \lambda^{-1}\phi_{k_{n+2}}(f_1 f_{n+2})$$

$$= e_{n+2}$$

▨

Lem 4.6

$$\phi_{l_n}(x) := \lambda^{-1} \cancel{j_{n+2}}\, \underset{\overset{\cancel{\phi_{j_n}}\circ}{j_n^{-1}}}{}\phi_{j_{n+1}}(w_{n+2}^* k_{n+2}(x) w_{n+2}) \qquad x \in \mathcal{A}_{0,n+3}$$

$$\begin{tikzcd}
 & & \mathcal{A}_{1n} \arrow[d, "l_n"] \\
 & & \mathcal{A}_{0n+2} \arrow[d, "k_{n+2}"] \\
\mathcal{A}_{1n} \arrow[r, "j_n"'] & \mathcal{A}_{1n+1} \arrow[r, "j_{n+1}"'] & \mathcal{A}_{1n+2}
\end{tikzcd}$$

Proof. $= \lambda^{-1}\phi_{j_n}\phi_{j_{n+1}}\cdots\phi_{j_m}(w_{m+1}^* k_{n+2}(x, w_{m+1})$ $(m \geq n+1)$

Trivially, $\phi_{l_n} : \mathcal{A}_{0n+2} \to \mathcal{A}_{1n}$. ucp

$$\phi_{l_n}(l_n(x)) \qquad x \in \mathcal{A}_{1n}$$

$$= \lambda^{-1}\overset{j_n^{-1}}{\cancel{\phi_{j_n}}}\phi_{j_{n+1}}(w_{n+2}^* k_{n+2}(l_n(x)) w_{n+2})$$

$$= \lambda^{-2}\cancel{\phi_{j_n}}\phi_{j_{n+1}}(w_{n+2}^* \underset{f_1}{\uparrow} E_{k_{n+2}}(\underset{f_1}{\uparrow}\Psi_1(x)) w_{n+2})$$

$$= \lambda^{-1}\cancel{\phi_{j_n}}\phi_{j_{n+1}}(w_{n+2}^* f_1 \Psi_1(x) w_{n+2}) \qquad \text{vertical Markov}$$

$$= \lambda^{-1}\cancel{\phi_{j_n}}\phi_{j_{n+1}}(f_{n+2}\, x)$$

$$= x$$

▨

Lem. 4.7

$$\begin{tikzcd} \mathcal{A}_{1n} \arrow[r, "j_n"] \arrow[d, "l_n"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{A}_{1n+1} \arrow[d, "l_{n+1}"] \\ \mathcal{A}_{0n+2} \arrow[r, "i_{n+2}"'] & \mathcal{A}_{0n+3} \end{tikzcd}$$

$(n \geq -1)$

comm. sq.

<u>proof.</u>

$x \in \mathcal{A}_{0,n+2}$

$\phi_{l_{n+1}}(i_{n+2}(x))$

$= \lambda^{-1} \phi_{j_{n+1}} \phi_{j_{n+2}} (w_{n+3}^{*} k_{n+3} i_{n+2}(x) \, w_{n+3})$

$= \lambda^{-1} \phi_{j_{n+1}} \phi_{j_{n+2}} (\underbrace{w_{n+3}^{*} \, j_{n+2} k_{n+2}(x) \, w_{n+3}}_{\substack{\shortparallel \\ \frac{1}{\lambda} E_{j_{n+1}}(w_{n+2}^{*} k_{n+2}(x) w_{n+2}) e_{n+3}}})$

$= \lambda^{-1} \phi_{j_{n+1}} (w_{n+2}^{*} k_{n+2}(x) \, w_{n+2})$

$\in \mathrm{Im}(j_n)$

☆ $\lambda^{-1} \phi_{j_{n+1}} (~~w_{n+2}^{*}~~ f_{n+2} \overset{\mathcal{A}_{1,n+2}}{\overset{\Psi}{y}} f_{n+2}) \cdot f_{n+2}$

horizontal Markov.

$= f_{n+2} \, y \, f_{n+2} \cdot f_{n+2}$

$= f_{n+2} \cdot \lambda^{-1} \phi_{j_{n+1}} (f_{n+2} \, y \, f_{n+2})$

$\Rightarrow \lambda^{-1} \phi_{j_{n+1}} (f_{n+2} \, y \, f_{n+2}) \in \{f_{n+2}\}' \cap \underset{\substack{\shortparallel \\ \mathrm{Im}(j_n)}}{\mathcal{A}_{1,n+1}}$

よって $\phi_{l_n} = \lambda^{-1} \, \underset{\sim\sim}{j_n^{-1}} \circ \phi_{j_{n+1}} (w_{n+2}^{*} k_{n+2}(\ \ ) \, w_{n+2})$.

これは well-defined.

☆ $l_{n+1} \cdot j_n (\overset{\mathcal{A}_{1n}.}{\overset{\Psi}{x}}) = \lambda^{-1} \phi_{k_{n+3}} (\Phi_1 (j_n(x)))$

$= \lambda^{-1} \phi_{k_{n+3}} (w_{n+3} \, j_n(x) \, w_{n+3}^{*})$

$= \lambda^{-1} \phi_{k_{n+3}} (w_{n+2} \, x \, w_{n+2}^{*})$

~~$= \lambda^{-1} \lambda^{-1} \phi_{k_{n+3}} (f_1 k_{n+2}(v_{n+2}) \cdot$~~

comm sq.

$= \lambda^{-1} \phi_{k_{n+2}} (w_{n+2} \, x \, w_{n+2}^{*})$

$= l_n(x)$

$$\begin{tikzcd} & & \mathcal{A}_{0n+2} \arrow[r] \arrow[d] & \mathcal{A}_{0n+3} \arrow[d, "k_{n+3}"] \\ \mathcal{A}_{1n} \arrow[r] & \mathcal{A}_{1n+1} \arrow[r] & \mathcal{A}_{1n+2} \arrow[r] & \mathcal{A}_{1n+3} \end{tikzcd}$$

$\mathcal{A}_{1n} \ni x$, $\Phi_1(x)$

§4.4 Crossed products

$$\mathcal{B}_0 := \mathcal{A}_0 \rtimes_{\Phi_0} \mathbb{N} \ni S_0$$

$$\mathcal{B}_1 := \mathcal{A}_1 \rtimes_{\Phi_1} \mathbb{N} \ni S_1$$

Lem. 4.8

- $\phi_{\Phi_0}(x) := \lambda^{-1}\phi_{\mathrm{i}n+1}(v_{n+2}^* x v_{n+2})$ $\quad x \in \mathcal{A}_{0,n+2}$.

- $\phi_{\Phi_1}(y) = \lambda^{-1}\phi_{\mathrm{j}n+1}(w_{n+2}^* y w_{n+2})$ $\quad y \in \mathcal{A}_{1,n+2}$

ϕ_{Φ_k} extends to a left inv. of Φ_k on $\mathcal{A}_k$

$\lrcorner$

Proof.

For $x \in \mathcal{A}_{0,n+2}$,

$$\lambda^{-1}\phi_{\mathrm{i}n+2}(v_{n+3}^* x v_{n+3})$$
$$= \lambda^{-1}\phi_{\mathrm{i}n+2}\left(\frac{1}{\lambda} e_{n+3} v_{n+2}^* x v_{n+2} e_{n+3}\right)$$
$$= \lambda^{-1}\lambda^{-1}\phi_{\mathrm{i}n+2}\left(E_{\mathrm{i}n+1}(v_{n+2}^* x v_{n+2}) e_{n+3}\right)$$
$$= \lambda^{-1}E_{\mathrm{i}n+1}(v_{n+2}^* x v_{n+2})$$

より ϕ_{Φ_0} extends to $\mathcal{A}_0$.

It is trivial $\phi_{\Phi_0}\cdot\Phi_0 = \mathrm{id}$

Lem. 4.9

$$\phi_{\Phi_0}(x) = S_0^* x S_0 \qquad x \in \mathcal{A}_0$$

$$\phi_{\Phi_1}(y) = S_1^* y S_1 \qquad y \in \mathcal{A}_1$$

$\lrcorner$

$x \in \mathcal{A}_{0,n+2}$. $\rightarrow$ $\phi_{\Phi_0}(x) \in \mathcal{A}_{0,n+1} \cap \{e_{n+2}\}' = \mathcal{A}_{0,n}$

$$\left(\phi_{\Phi_0}(x) e_{n+2} = \cancel{\phi_{\Phi_0}(x} \, S_0^* x \underset{\underset{e_{n+4}\,\cancel{e_2}}{\parallel}}{\Phi_0(e_{n+2})} S_0 = e_{n+2}\phi_{\Phi_0}(x) \right)$$

Proof.

Enough to show

$$\Phi_0 \cdot \phi_{\Phi_0}(x) = e_2 x e_2 \qquad x \in A_0$$

$$\Big(\rightsquigarrow \underset{\overset{\shortparallel}{\phi_{\Phi_0}(x)}}{S_0^* \Phi_0(\phi_{\Phi_0}(x)) S_0} = \underset{\overset{\shortparallel}{S_0^* x S_0}}{S_0^* e_2 x e_2 S_0} \Big)$$

$x \in A_{0,n+2}$

$$\Phi_0 \cdot \phi_{\Phi_0}(x) = \Phi_0 (\lambda^{-1} \phi_{i_{n+1}} (v_{n+2}^* x \, v_{n+2}))$$

$$= \lambda^{-1} v_{n+3} \overset{e_{n+3}}{\phi}_{i_{n+1}} (\underset{\uparrow \, e_{n+2}}{v_{n+2}^*} x \, v_{n+2}) \, v_{n+3}^*$$

hor. Markov

$$= \Phi \Big(\frac{v_{n+2}}{\sqrt{\lambda}} \underset{\otimes e_{n+2}}{\cdot} v_{n+2}^* x \, v_{n+2} \cdot v_{n+3}^* \Big)$$

$$= \frac{X^*}{\sqrt{\lambda}} e_2 x \sqrt{\lambda} e_2$$

$$= e_2 x e_2 .$$

$$v_{n+1} v_n^* = \frac{1}{\sqrt{\lambda}^{n-1}} \cdot \frac{1}{\sqrt{\lambda}^{n-2}} e_2 \cdots e_n e_{n+1} e_n \cdots e_2$$

$$= \sqrt{\lambda} \, v_{n+1} v_{n+1}^*$$

$$= \sqrt{\lambda} \, e_2$$

Recall

$$\begin{tikzcd}
A_{0n} \arrow[r] \arrow[d, "k_n"'] \arrow[dr, phantom, "\circlearrowleft"] & A_{0n+1} \arrow[d, "k_{n+1}"] & & A_0 \arrow[d, "k"] \\
A_{1n} \arrow[r] \arrow[d, "l_n"'] \arrow[dr, phantom, "\circlearrowright"] & A_{1n+1} \arrow[d, "l_{n+1}"] \arrow[r, phantom, "\rightsquigarrow"] & & A_1 \arrow[d, "l"] \\
A_{0n+2} \arrow[r] & A_{0n+3} & & A_0
\end{tikzcd}$$

$k(v_n) = w_2^* w_n \qquad w_n = w_2 \, k(v_n)$

$l(w_n) = v_3^* v_{n+2} \qquad v_{n+2} = v_3 \, l(w_n)$

$l \, k(v_n) =$

Lem. 4.10

k & l extends to

$k : \mathcal{B}_0 \to \mathcal{B}_1 \qquad k(S_0) = w_2^* S_1$

$l : \mathcal{B}_1 \to \mathcal{B}_0. \qquad l(S_1) = v_3^* S_0$

Proof. $\quad x \in \mathcal{A}_{0,n}$

- $$w_2^* S_1 k_n(x) = w_2^* \Xi_1(k_n(x)) S_1$$
$$= \underbrace{w_2^* w_{n+2}}\, k_n(x)\, w_{n+2}^* S_1$$

$$\left(\begin{aligned} w_2^* w_n &= \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\sqrt{\lambda}^{n-1}} f_2 f_1 f_1 \cdots f_n \\ &= \frac{\lambda}{\sqrt{\lambda}^n} f_2 \cdots f_n \\ &= k_n(v_n). \qquad w_n = w_2 k_n(v_n) \end{aligned} \right)$$

$$= k_{n+2}(v_{n+2} x v_{n+2}^*) w_2^* S_1.$$

$$(w_2^* S_1)^* w_2^* S_1 = S_1^* w_2 w_2^* S_1$$
$$= \phi_{\Xi_1}(f_1)$$
$$= S_1^* f_1 S_1$$
$$= 1.$$

- $$v_3^* v_n = \frac{1}{\sqrt{\lambda}} \frac{1}{\sqrt{\lambda}^{n-2}} e_3 e_2 \cdot e_2 e_3 \cdots e_n$$
$$= \frac{\lambda}{\sqrt{\lambda}^{n-1}} e_3 \cdots e_n$$
$$= \frac{\lambda}{\sqrt{\lambda}^{n-1}} l_{n-2}(f_1 \cdots f_{n-2}) \qquad \frac{1}{\sqrt{\lambda}^{n-3}}$$
$$= l_{n-2}(w_{n-2})$$

- $$v_3^* S_0\, l(\overset{\mathcal{A}_{1n}}{\overset{\downarrow}{x}}) = v_3^* \Xi_0(l(x)) S_0$$
$$= v_3^* v_{n+4}\, l(x)\, v_{n+4}^* S_0$$
$$= l(w_{n+2} x w_{n+2}^*) v_3^* S_0$$
$$= l(\Xi_1(x)) v_3^* S_0$$

- $$(v_3^* S_0)^* v_3^* S_0 = S_0^* \underset{e_2}{v_3 v_3^*} S_0 = \phi_{\Xi_0}(e_2) = S_0^* e_2 S_0$$
$$= 1.$$

$\square$

Lem.4.11

$S_0 \in \mathrm{Mor}(\mathrm{id}_{B_0}, \ell k)$

$S_1 \in \mathrm{Mor}(\mathrm{id}_{B_1}, k\ell)$

Proof.

- $x \in A_{0n}$

$\ell k(x) S_0$

$= \lambda^{-1} \phi_{k_{n+2}}(\Psi_1(k_n(x))) \overset{e_2}{\downarrow} S_0$

$= \lambda^{-1} \phi_{k_{n+2}}(w_{n+2} k_n(x) w_{n+2}^*) S_0$

$= \lambda^{-1} \phi_{k_{n+2}}(w_{n+2} k_n(x) \underbrace{w_{n+2}^* f_2}_{\lambda k(v_{n+2}^* f_2 \frac{1}{\sqrt{\lambda}}}) S_0$ 外1. e_2

$= \cancel{\lambda^{-1}} \phi_{k_{n+2}}(w_{n+2}) x v_{n+2}^* S_0 \frac{1}{\sqrt{\lambda}}$

$= \phi_{k_{n+2}}(f_1) \cdot \frac{1}{\sqrt{\lambda}^{n+1}} e_2 \cdots e_{n+2} x v_{n+2}^* S_0 \frac{1}{\sqrt{\lambda}}$

$= v_{n+2} x v_{n+2}^* S_0.$

$\ell k(x) S_0 = \Psi_0(x) S_0 = S_0 x.$

- $S_0 S_0$

$\ell k(S_0) S_0 = \ell(w_2^* S_1) S_0$

$= \ell(w_2^*) v_3^* S_0 S_0$

$= v_4^* \underbrace{\cancel{v_3 v_3^*}}_{\cancel{e_2}} S_0^2$

$= \frac{1}{\sqrt{\lambda}^2} e_4 e_3 e_2 S_0^2$

$$\left(\begin{aligned}\Psi_0(e_2) &= v_4 e_2 v_4^* \\ &= \frac{1}{\sqrt{\lambda}^2} e_2 e_3 e_4\, e_2\, e_4 e_3 e_2 \frac{1}{\sqrt{\lambda}^2} \\ &= \frac{1}{\lambda^2} \lambda\, e_2 e_4 e_3 e_2\end{aligned}\right)$$

$= e_2 e_4$

$= \frac{1}{\lambda} e_4 e_3 e_2 \Psi_0(e_2) S_0^2$

$= \frac{1}{\lambda} e_4 e_3 e_2 e_2 \textcircled{e_4} S_0^2$

$= e_4 e_2 S_0^2$

$= S_0^2$

· $x \in \cancel{B_{1n}}\ A_{1n}$.

$k\ell(x)\, S_1$

$= k(\lambda^{-1}\phi_{k_{n+2}}(\Phi_1(x)))\ S_1$

$= \lambda^{-1}\ k\,\phi_{k_{n+2}}(w_{n+2}\, x\, w_{n+2}^{*}\overset{f_1}{\underset{\uparrow f_1}{)}})\, S_1$

vertical Markov

$= w_{n+2}\, x\, w_{n+2}^{*}\, S_1$

$= \Phi_1(x)\, S_1$

$= S_1\, x$.

$k\ell(S_1)\, S_1$

$= k(v_3^{*} S_0)\, S_1$

$= k(v_3^{*})\, w_2^{*}\, S_1\, S_1$

$= w_3^{*}\, f_1\, S_1\, S_1$

$= \lambda^{-1}\, f_3 f_2 f_1\, S_1^{2}$

$= \lambda^{-1}\, f_3 f_2 f_1\, \Phi_1(f_1)\, S_1^{2}$

$$\left(\begin{aligned} \Phi_1(f_1) &= w_3\, f_1\, w_3^{*} \\ &= \frac{1}{\lambda^2}\, f_1 f_2 f_3 \enclose{circle}{f_1} f_3 f_2 f_1 \\ &= f_1 f_3 \end{aligned}\right)$$

$= \lambda^{-1}\, f_2\, f_2 f_1\, f_3\, S_1^{2}$

$= f_1 f_3\, S_1^{2}$

$= S_1^{2}$.

Lem. 4.12

$$(\ell k)^n(S_0) = v_{2n+2}^* S_0$$

$$(\ell k)^n \ell(S_1) = v_{2n+3}^* S_0$$

$$(k\ell)^n k(S_0) = w_{2n+2}^* S_1$$

$$(k\ell)^n(S_1) = w_{2n+1}^* S_1$$

Proof.

$$\ell k(S_0) = v_4^* S_0$$

$$(\ell k)^{n+1}(S_0) = \ell k(v_{2n+2}^* S_0)$$

$$= \ell(w_2^* w_{2n+2})^* v_4^* S_0$$

$$= (v_4^* v_3 \cdot v_3^* v_{2n+4})^* v_4^* S_0$$

$$= \cancel{v_4^*} v_{2n+4}^* \cancel{v_4^*} S_0$$

~~$v_4^* v_{2n+4} v_4^* = \frac{1}{\sqrt{\lambda}^2} \cdot \frac{1}{\sqrt{\lambda}^{2n+2}} e_4 e_3 e_2 \cdot e_2 e_3 \cdots e_{2n+4} \, v_4^*$~~

~~$= \frac{1}{\lambda^n} e_4 \cdots e_{2n+4} \, v_4^*$~~

~~$= \frac{1}{\lambda^n} e_4 e_5 v_4^* e_6 \cdots e_{2n+4}$~~

~~$= \frac{1}{\lambda^n} \frac{1}{\sqrt{\lambda}^2} e_4 e_3 e_2$~~

$$\ell(S_1) = v_3^* S_0$$

$$(\ell k)^{n+1} \ell(S_1) = \ell k(v_{2n+3}^* S_0) = \ell k(v_{2n+3}^*) v_4^* S_0$$

$$= v_{2n+5}^* \cancel{v_4 v_4^*} S_0$$

$$(k\ell)^n k(S_0) = k(v_{2n+2}^* S_0) = w_{2n+2}^* \cancel{w_2 w_2^*} S_1$$

$$(k\ell)^n(S_1) = k(v_{2n+1}^* S_0) = w_{2n+1}^* \cancel{w_2 w_2^*} S_1$$

§4.5 Std λ-lattice

$$\begin{tikzcd}
 & & & A_{00} \arrow[r] \arrow[d, "k_0"] & A_{01} \arrow[r] \arrow[d] & A_{02} \arrow[r] \arrow[d] & {} \\
 & & A_{1-1} \arrow[r] \arrow[d, "l_{-1}"] & A_{10} \arrow[r] \arrow[d, "l_0"] & A_{11} \arrow[r] \arrow[d] & A_{12} \arrow[r] \arrow[d] & {} \\
 & A_{00} \arrow[r] \arrow[d, "k_0"] & A_{01} \arrow[r] \arrow[d, "k_1"] & A_{02} \arrow[r] \arrow[d, "k_2"] & A_{03} \arrow[r] \arrow[d] & A_{04} \arrow[r] \arrow[d] & \cdot \\
A_{1-1} \arrow[r] & A_{10} \arrow[r] & A_{11} \arrow[r] & A_{12} \arrow[r] & A_{13} \arrow[r] & A_{14} &
\end{tikzcd}$$

NOTATION

$$A_{i\cdot j} = A_{i+2,j-2}$$

$$A_{i\cdot j} = A_{i-2,j+2} \qquad i+j = (i-2)+(j+2)$$

$$\begin{tikzcd}
 & & 00 \arrow[r, dash] \arrow[d, dash] & 01 \arrow[r, dash] \arrow[d, dash] & 02 \arrow[r, dash] \arrow[d, dash] & {} \\
 & 1{-}1 \arrow[r, dash] \arrow[d, dash] & 10 \arrow[r] \arrow[d, dash] & 11 \arrow[r] \arrow[d, dash] & 12 \arrow[r, dash] \arrow[d, dash] & {} \\
2{-}2 \arrow[r, dash] \arrow[d, dash] & 2{-}1 \arrow[r, dash] \arrow[d, dash] & 20 \arrow[r, dash] \arrow[d, dash] & 21 \arrow[r, dash] \arrow[d, dash] & 22 \arrow[r, dash] \arrow[d, dash] & {} \\
{} & {} & {} & {} & {} &
\end{tikzcd}$$

Defn. 4.13

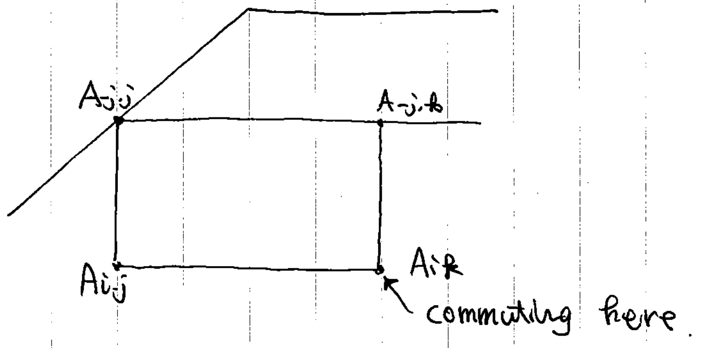
A λ-lattice (A_{ij}) is standard

if $\forall i \geqq 0.\ \forall j \leqq 0$

$\forall k \geqq -j$

$$[A_{i\cdot j},\ A_{-j,k}] = 0 \quad \text{in } A_{i,k}$$

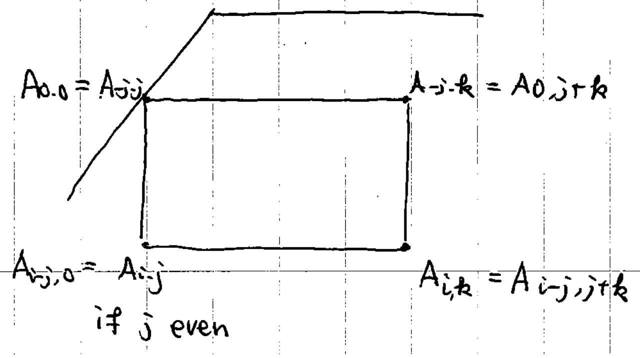
☆



☆ λ-lattice std ⟺ $[A_{i\cdot j},\ A_{-j\cdot k}] = 0$ in $A_{i\cdot k}$

$\forall i \geqq 0 \quad \forall j = 0, -1$

$\forall k \geqq -j$



etc.

Lem. 4.14 For $n \geq 0$.

$$\mathcal{A}_{0,2n} \subset {}_{B_0}\mathrm{Mor}_{B_0}((\ell k)^n, (\ell k)^n)$$

$$\mathcal{A}_{0,2n+1} \subset {}_{B_0}\mathrm{Mor}_{B_0}((\ell k)^n \ell, (\ell k)^n \ell)$$

$$\mathcal{A}_{1,2n} \subset {}_{B_1}\mathrm{Mor}_{B_0}((k\ell)^n k, (k\ell)^n k)$$

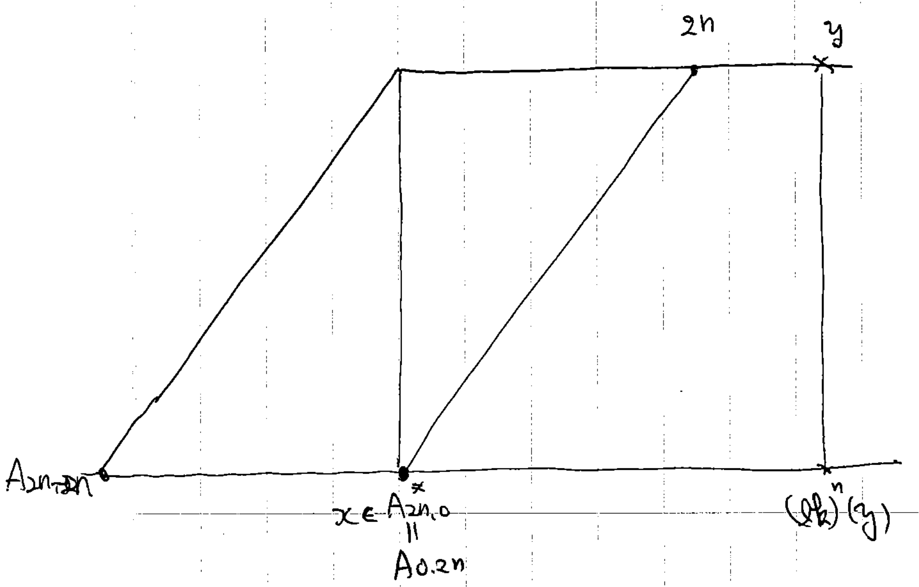
$$\mathcal{A}_{1,2n-1} \subset {}_{B_1}\mathrm{Mor}_{B_1}((k\ell)^n, (k\ell)^n)$$

$\lrcorner$

Proof

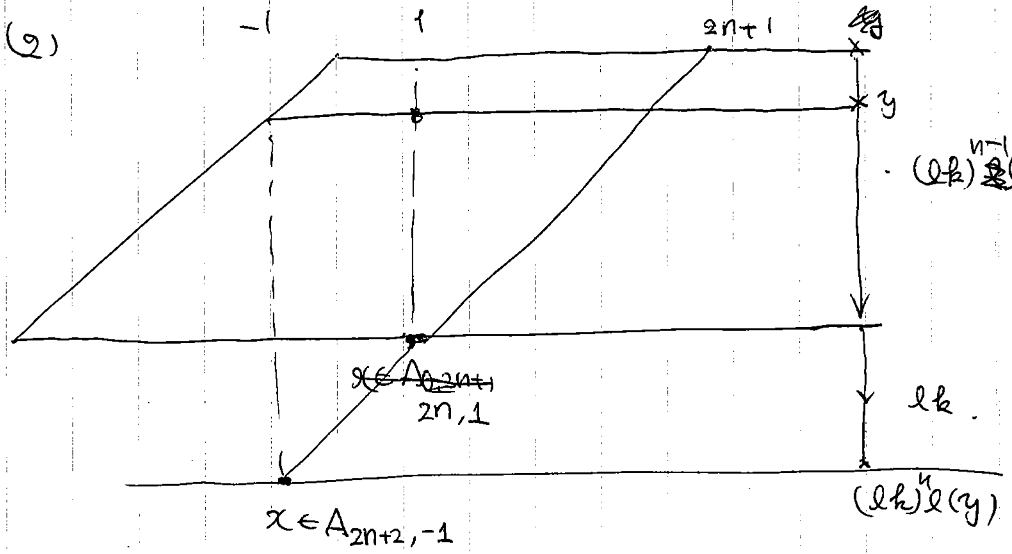
(1) ~~$x \in \mathcal{A}_{0,m}$ $(\ell k)^n(x) \in \mathcal{A}_{0,m+2n}$~~

~~$\ell k(x) = \lambda^{-1}\phi_{k_{m+2}}(\Psi_1(kx))$~~

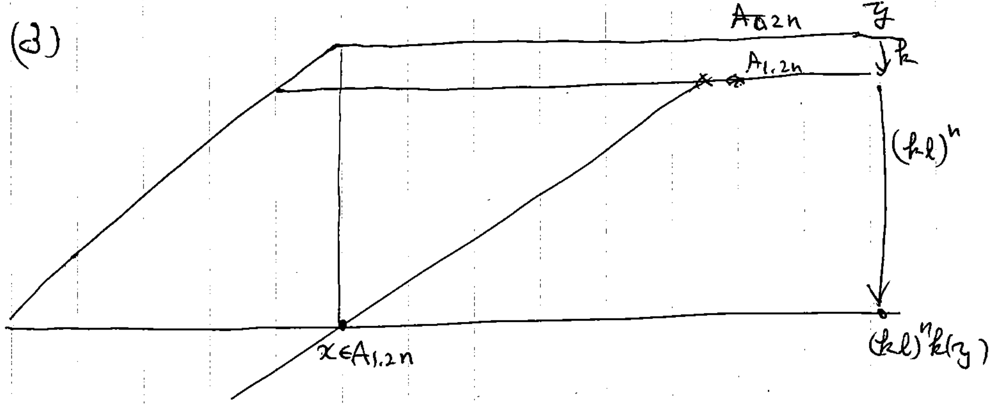


Std. より $[x, (\ell k)^n(y)] = 0$.

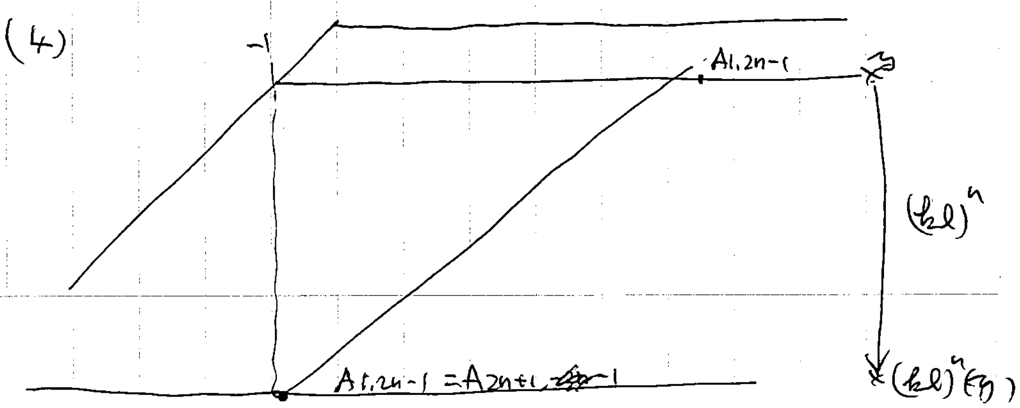
(2)



(3)



(4)



(1) $(lk)^n(S_0) = v_{2n+2}^* S_0$

$x \in A_{0,2n}$

$$v_{2n+2}^* S_0 x = v_{2n+2}^* lk(x) S_0$$

$$= v_{2n+2}^* \Phi_0(x) S_0$$

$$= v_{2n+2}^* v_{2n+2} x v_{2n+2}^* S_0$$

$$= \underset{\curvearrowright}{e_{2n+2}\, x}\ v_{2n+2}^* S_0$$

$$= x v_{2n+2}^* S_0.$$

(2) $(lk)^n l(S_1) \overset{A_{0,2n+1}\ni}{x} = v_{2n+3}^* S_0 x$

$$= v_{2n+3}^* \Phi_0(x) S_0$$

$$= \overset{\curvearrowright}{e_{2n+3}\, x}\ v_{2n+3}^* S_0$$

$$= x (lk)^n l(S_1)$$

(3) $(kl)^n k(S'_0) \overset{A_{1,2n}\ni}{x} = ~~(kl)^n f~~\ w_{2n+2}^* S_1 x$

$$= w_{2n+2}^* \Phi_1(x) S_1$$

$$= \overset{\curvearrowright}{f_{2n+2}\, x}\ w_{2n+2}^* S_1$$

$$= x (kl)^n k(S'_0)$$

(4) $(kl)^n(S'_1) \overset{A_{1,2n-1}\ni}{x} = w_{2n+1}^* S_1 x$

$$= w_{2n+1}^* \Phi_1(x) S_1$$

$$= x (kl)^n (S'_1).$$

3

$A_0^{alg} := \bigcup_{n \geq 0} A_{0n} \subset A_0 \quad \Psi_0$

$A_1^{alg} := \bigcup_{n \geq -1} A_{1n} \subset A_1 \quad \Psi_1$ $-$inv.

Lem. 4.15

- $A_0^{alg} \cap {}_{B_0}\mathrm{Mor}_{B_0}((\ell k)^r, (\ell k)^s) = \delta_{r,s} A_{0,2r}$
- $A_0^{alg} \cap {}_{B_0}\mathrm{Mor}_{B_1}((\ell k)^r \ell, (\ell k)^s \ell) = \delta_{r,s} A_{0,2r+1}$
- $A_1^{alg} \cap {}_{B_1}\mathrm{Mor}_{B_1}((k\ell)^r k, (k\ell)^s k) = \delta_{r,s} A_{1,2r}$
- $A_1^{alg} \cap {}_{B_1}\mathrm{Mor}{}_{B_1}((k\ell)^r, (k\ell)^s) = \delta_{r,s} A_{1,2r-1}$

Proof.

(1) $x \in$ LHS.

Suppose $r \gneqq s$.

$$x(\ell k)^r(e_t) = (\ell k)^s(e_t)x$$
$$\| \qquad\qquad \|$$
$$x e_{t+2r} \qquad e_{t+2s} x$$
$$(\forall t \geq 2)$$

$x \in A_{0,2n}$ とする.

t を $\underset{t+2r}{\overset{\wedge}{t+2s}} > 2n$ となるようにとると.

$$E_{A_{0,t+2s}}(x e_{t+2r}) = E_{A_{0,t+2s}}(x e_{t+2s})$$
$$\| \qquad\qquad \|$$
$$x \cdot \lambda. \qquad\qquad x e_{t+2s}$$

~~2より $E_{A_{0,2n}}$.~~

$\to \quad (1-\lambda)\, x\, e_{t+2s} = 0$

$t+2s \geq 2n+2$ ととると.

$\to \quad E_{A_{0,2n}} \quad x^* x = 0 \qquad x = 0.$

Suppose $r = s$.

$$\begin{cases} x e_t = e_t x & \forall t \geq 2r+2. \\ x \in A_{0,2n} \end{cases}$$

If $n \leq r \to 2n \leqq 2r \leq 2r+2$. OK. $\to x \in A_{0,2n} \subset A_{0,2r}$.

If $n > r \to 2n \geqq 2r+2$. $\to t$ として $t = 2n+1$ をとれる.

$\to \quad x \in A_{0,2n} \cap \{e_{2n+1}\}' = A_{0,2n-1}$

$2n = 2r+2 \to 2n-1 = 2r+1$. $x \in A_{0,2r+1} \xrightarrow[t=2r+2]{} x \in A_{0,2r}$

$2n-1 \gneqq 2r+1 \to t = e_{2n} \to x \in A_{0,2n-2}$. $\underset{くり返す}{\leadsto} x \in A_{0,2r}$

(4) 同様

(2). $x \in LHS$

$k(x) \in {}_{B_1}\mathrm{Mor}_{B_1}((kl)^{r+1}, (kl)^{s+1})$

$\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad \delta_{r,s}\ \mathcal{A}_{1,2(r+1)-1}$

$\to r \neq s \Rightarrow x = 0.$

$r = s.$

$k(x) = y \in \mathcal{A}_{1,2r+1}$

$lk(x) = l(y) \in \mathcal{A}_{0,2r+3}$

$\to \quad x = S_0^* lk(x) S_0 = S_0^* l(y) S_0$

$\in S_0^* \mathcal{A}_{0,2r+3} S_0$

$\subset \mathcal{A}_{0,2r+1}.$

(3) $x \in LHS$

$l(x) \in {}_{B_0}\mathrm{Mor}_{B_0}((lk)^{r+1}, (lk)^{s+1})$

$\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad \delta_{r,s}\ \mathcal{A}_{0,2r+2}.$

$\to r \neq s \quad \to x = 0$

$r = s$ なら

$kl(x) \in \mathcal{A}_{1,2r+2}.$

$x \in S_1^* \mathcal{A}_{1,2r+2} S_1 \subset \mathcal{A}_{1,2r}.$

▨

$B_0^{alg} := \mathcal{A}_0^{alg} \vee \{S_0\}$ ~~if~~ necessari

$B_1^{alg} := \mathcal{A}_1^{alg} \vee \{S_1\}$

Lem. 4.16 $r, s \geq 0$.

(1) $B_0^{alg} \cap {}_{B_0}\mathrm{Mor}_{B_0}((lk)^r, (lk)^s)$

$$= \begin{cases} S_0^{*(r-s)} \mathcal{A}_{0,2r} & \text{if } r > s \\ \mathcal{A}_{0,2r} & \text{if } r = s \\ \mathcal{A}_{0,2s} S_0^{s-r} & \text{if } r < s \end{cases}$$

とくに → $r = s = 0$

$B_0^{alg} \cap {}_{B_0}\mathrm{Mor}_{B_0}(id, id) = \mathcal{A}_{0,0} = \mathbb{C}$,

(2) $B_0^{alg} \cap {}_{B_0}\mathrm{Mor}_{B_1}((lk)^r l, (lk)^s l)$

(~~2~~)
$$= \begin{cases} S_0^{*(r-s)} \mathcal{A}_{0,2r+1} & \text{if } r > s \\ \mathcal{A}_{0,2r+1} & \text{if } r = s \\ \mathcal{A}_{0,2s+1} S_0^{(s-r)} & \text{if } r < s \end{cases}$$

(3) $B_1^{alg} \cap {}_{B_1}\mathrm{Mor}_{B_0}((kl)^r k, (kl)^s k)$

$$= \begin{cases} S_1^{*(r-s)} \mathcal{A}_{1,2r} & \text{if } r > s \\ \mathcal{A}_{1,2r} & \text{if } r = s \\ \mathcal{A}_{1,2s} S_1^{(s-r)} & \text{if } r < s \end{cases}$$

(4) $B_\phi^{alg} \cap {}_{B_\phi}\mathrm{Mor}_{B_1}((kl)^r, (kl)^s)$

$$= \begin{cases} S_1^{*(r-s)} \mathcal{A}_{\phi,2r-1} & \text{if } r > s \\ \mathcal{A}_{1,2r-1} & \text{if } r = s \\ \mathcal{A}_{1,2s-1} S_1^{*(s-r)} & \text{if } r < s \end{cases}$$

Proof.

(1) ~~($r > s$ のとき.)~~

$x \in$ LHS.

WMA

$$x = \begin{cases} S_0^{*t} a & (t \geq 1). \quad a \in \mathcal{A}_0^{alg} \\ a & \\ a S_0^{*} & (t \geq 1) \quad a \in \mathcal{A}_0^{alg}. \end{cases}$$

Since the gauge actions γ_0, γ_1 commute with k & l.

$x = S_0^{*t} a$ のとき. (WMA. $\Psi_0^t(1) a = a$).

$a = S_0^{t} x : (lk)^r \to (lk)^{2t+s}$

前 Lem より. $r \neq 2t+s$ のときは. $a = 0$.

(i.e. $r - s > 0$ かつ. $r-s$: even).

$a \in \delta_{r,2t+s} \mathcal{A}_{0,2r}$

よって. $\sum_{t \geq 1} \delta_{r,2t+s} S_0^{*t} \mathcal{A}_{0,2r}$

$x = a \in \mathcal{A}_{0,2r} \cdot \delta_{r,s}$

$x = a S_0^{t} \in \sum_{t \geq 1} \delta_{s,t+s} \mathcal{A}_{0,2s} S_0^{t}$

よって

$\mathcal{B}_0 \cap {}_{B_0}\mathrm{Mor}_{B_0}((\ell k)^r, (\ell k)^s)$

$$= \sum_{t \geq 1} \delta_{r,t+s} S_0^{*t} \mathcal{A}_{0,2r} + \mathcal{A}_{0,2r} \delta_{r,s} + \sum_{t \geq 1} \delta_{s,t+r} \mathcal{A}_{0,2s} S_0^{t}$$

$$= \begin{cases} S_0^{*(r-s)} \mathcal{A}_{0,2r} & r > s \\ \mathcal{A}_{0,2r} & r = s \\ \mathcal{A}_{0,2s} S_0^{s-r} & r < s \end{cases}$$

(2) $x = \sum_{t \geq 1} S_0^{*t} b_t + b_0 + \sum_{t \geq 1} c_t S_1^{t}$

$S_0^{*t} b_t \in \mathrm{Mor}((\ell k)^r \ell, (\ell k)^s \ell)$

$b_t \in (\ell k)^r \ell \longrightarrow (\ell k)^{s+t} \ell \in \delta_{r,t+s} \mathcal{A}_{0,2r+1}$

$b_0 : (\ell k)^r \ell \longrightarrow (\ell k)^s \ell \in \delta_{r,s} \mathcal{A}_{0,2r+1}$

$c_t : (\ell k)^s \ell \longleftarrow (\ell k)^{r+t} \ell \in \delta_{s,t+r} \mathcal{A}_{0,2s+1}$

(2) $x = \sum_{t \geq 1} S_1^{*t} b_t + b_0 + \sum_{t \geq 1} c_t S_1^{t}$.

$S_1^{*t} b_t \in \mathrm{Mor}((k\ell)^r, (k\ell)^s)$

$\rightsquigarrow b_t \in (k\ell)^r \longrightarrow (k\ell)^{t+s}$

$\in \delta_{r,t+s} \mathcal{A}_{1,2r-1}$.

$b_0 \in \mathcal{A}_{1,2r} \delta_{r,s}$

$c_t \in \delta_{s,t+r} \mathcal{A}_{1,2s-1}$.

§4.6 Std lattice $\xrightarrow[\S 4.1.]{②}$ C^*-2-category.

$$\mathcal{D} := \begin{pmatrix} \mathcal{D}_{00} & \mathcal{D}_{01} \\ \mathcal{D}_{10} & \mathcal{D}_{11} \end{pmatrix}$$ C^*-2-category

(idemp. comp は 仮定しない)

$\mathcal{D}_{00}$ objs : $(\ell k)^n$ $n \geq 0$.

Mor : interwiners in $B_0 \otimes \mathcal{O}_2$
↑ if necessary, direct sum.

$\mathcal{D}_{01}$ objs : $(\ell k)^n \ell$ $n \geq 0$

Mor : ―― $\sim B_0$

$\mathcal{D}_{10}$ objs : $(k\ell)^n k$ $n \geq 0$

Mor : ―― $\sim B_1$

$\mathcal{D}_{11}$ objs : $(k\ell)^n$ $n \geq 0$

Mor : ―― $\sim B_1$

$\mathcal{D}_{rs} \times \mathcal{D}_{st} \longrightarrow \mathcal{D}_{rt}$ bilinear functor:

obj $(\alpha, \beta) \longmapsto \alpha \cdot \beta$ composed ~~from~~.

$(T, S) \longmapsto T \times S$. product. morphisms.

$\mathcal{D}$ の subcategory $\mathcal{C}$ をつくる

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & \mathcal{C}_{01} \\ \mathcal{C}_{10} & \mathcal{C}_{11} \end{pmatrix}$$

$\mathcal{C}_{rs}$: objs $\mathcal{D}_{rs}$ と同じ.

$\mathcal{C}_{rs}(\alpha, \beta) := \underset{\cancel{A^{alg}}\ B_r^{alg}}{\cancel{B^{alg}}} \cap \mathcal{D}_{rs}(\alpha, \beta)$.

$\mathcal{C}_{rs} \times \mathcal{C}_{st} \longrightarrow \mathcal{C}_{rt}$ bilinear functor.

は $\mathcal{D}$ のものを 制限する.

$(T, S) \longmapsto T \times S$ ↖ $\cancel{A^{alg}}\ B_r^{alg}$ から出ないので.

well-defined.

$(\mathcal{O}_2$ テンソル$)$

こうして. C^*-2-category $\mathcal{C}$ を得る.

rigid であることをかくにんする.

<u>Lem</u> 4.17 (S_0, S_1) は (k, l) の conj. eq. をみたす: しかも std. sol. である.

$$S_1^* k(S_0) = \sqrt{\lambda}$$
$$S_0^* l(S_1) = \sqrt{\lambda}.$$

<u>Proof</u>.

$$S_1^* k(S_0) = S_1^* w_2^* S_1 \qquad w_2 = \frac{1}{\sqrt{\lambda}} f_1 f_2$$
$$= \underset{\overset{\nearrow}{S_1}}{S_1^*} \frac{1}{\sqrt{\lambda}} f_2 f_1 S_1$$
$$= \frac{\lambda}{\sqrt{\lambda}} S_1^* S_1$$
$$= \sqrt{\lambda}$$

$$S_0^* l(S_1) = S_0^* v_3^* S_0 \qquad v_3 = \frac{1}{\sqrt{\lambda}} e_2 e_3$$
$$= \underset{\overset{\nearrow}{e_2}}{S_0^*} \frac{1}{\sqrt{\lambda}} e_3 e_2 S_0$$
$$= \sqrt{\lambda}.$$

次に std. sol. であることを示す.

$x \in (l, l) = A_{0,1}$. として.

$$\begin{array}{ccc} S_0^*(x \otimes 1) S_0 & = & S_1^*(1 \otimes x) S_1 \\ \| \text{ i.e.} & & \| \\ S_0^* x S_0 & & S_1^* k(x) S_1 \end{array}$$ を示す.

$\mathbb{C} \ni$

$$S_0^* x S_0 = \Phi_{E_0}(x) \underset{\substack{\uparrow \\ \Phi_{E_0} \text{ の defn.}}}{=} \phi_{j_0}(x) = \tau(x).$$

$\cup$

$$S_1^* k(x) S_1 = \Phi_{E_1}(k(x)) = \lambda^{-1} \phi_{j_1}(w_2^* k(x) w_2)$$
$$= \lambda^{-1} \lambda^{-1} \phi_{j_1}(f_2 f_1 k(x) f_1 f_2)$$
$$= \lambda^{-2} \phi_{j_1}(\phi_{j_0}(f_1 k(x) f_1) f_2)$$
$$= \lambda^{-2} \phi_{j_0}(f_1 k(x) f_1) \cdot \lambda$$
$$= \lambda^{-1} \phi_{j_0}(f_1 k(x) f_1)$$

Ror. Markov,

$$\lambda^{-1} \phi_{j_0}(f_1 k(x) f_1) f_1 = f_1 \underbrace{k(x) f_1} = f_1 \frac{1}{\lambda} \phi_{j_0}(k(x) f_1) f_1$$
$$= \frac{1}{\lambda} \tau(k(x) f_1) f_1$$
$$= \frac{1}{\lambda} \tau(x \phi_k(f_1)) f_1 = \tau(x) f_1$$

よって $\mathcal{C}$ は rigid C^*-2-category である.

これを idemp. 完ビ化して. C^*-2-category $\underset{\substack{\nearrow \\ (\text{同じ記号})}}{\mathcal{C}}$ を得る. これは次の性質をもっている.

- pointed. object k & l. で生成される.

$(n \geq 0)$

$$\mathcal{C}_{00}((lk)^n, (lk)^n) = \mathcal{A}_{0,2n}$$

$$\mathcal{C}_{01}((lk)^n l, (lk)^n l) = \mathcal{A}_{0,2n+1}$$

$$\mathcal{C}_{10}((kl)^n k, (kl)^n k) = \mathcal{A}_{1,2n}$$

$$\mathcal{C}_{11}((kl)^n, (kl)^n) = \mathcal{A}_{1,2n-1}$$

$$\begin{tikzcd}
 & \mathcal{A}_{00} \arrow[r, "\cdot\otimes 1_l"] \arrow[d, "1_k\otimes\cdot"'] & \mathcal{A}_{01} \arrow[r, "\cdot\otimes 1_k"] \arrow[d, "1_k\otimes"'] & \mathcal{A}_{02} \arrow[r] \arrow[d, "1_k\otimes"'] & \cdots \\
\mathcal{A}_{1,-1} \arrow[r, "\cdot\otimes 1_k"'] & \mathcal{A}_{10} \arrow[r, "\cdot\otimes 1_l"'] & \mathcal{A}_{11} \arrow[r, "\cdot\otimes 1_k"'] & \mathcal{A}_{12} \arrow[r] & \cdots
\end{tikzcd}$$

以上により

$$\text{Std } \lambda \text{ lattice} \longmapsto \text{Pointed. } C^*\text{-2-category}$$

という関手を得る.

そこで $(0\lambda < 1)$ に対して 圏

$\mathbf{STL}_\lambda$

<u>Objs</u> std. λ lattices.

$$\mathbb{A} = (\mathcal{A}_{r,s}, k, l, i, \underset{\phi}{j}, \tau, e_n, f_n)$$

<u>Mor.</u>

Mor$(\mathbb{A}, \mathbb{B})$ consists of

$$\varphi = (\varphi_{rs} : \mathcal{A}_{rs} \to \mathcal{B}_{rs}) \text{ preserving } k.\underset{\phi}{l}.i.j.\tau.e_n.f_n.$$

φ は Jones proj を保つので.

$$\begin{tikzcd}
\mathcal{A}_0 \curvearrowleft \Phi_0 \arrow[r, bend left, "\varphi"] \arrow[d, shift right, "k"'] & \mathcal{B}_0 \curvearrowleft \Phi_0^B \arrow[d, shift right, "k"'] \\
\mathcal{A}_1 \curvearrowleft \Phi_1 \arrow[u, shift right, "l"'] \arrow[r, bend right, "\varphi"'] & \mathcal{B}_1 \curvearrowleft \Phi_1^B \arrow[u, shift right, "l"']
\end{tikzcd}$$

を保つ.

$$\begin{tikzcd}
\mathcal{C}_t := \mathcal{A}_t \rtimes_{\Phi_t} \mathbb{N} \arrow[r, "\varphi_t"] \arrow[d, "\wr"] & \mathcal{D}_t := \mathcal{B}_t \rtimes_{\Phi_t} \mathbb{N} \arrow[d, "\wr"] \\
\mathcal{C}_{k^A} & \mathcal{C}_{k^B} .
\end{tikzcd}$$

Object.

$\Phi_0 (l^A k^A)^n \longmapsto (l^B k^B)^n.$

Mor $\longrightarrow$ Mor.

より. 2圏 の 間 の functor を 定める:

$$\mathbb{S}\mathrm{td}_\lambda \longrightarrow C^*2\mathcal{C}\mathrm{at}_\lambda^{\mathrm{pt.}}$$ ~~を与う~~

これは 明らかに 圏同値 を 定める.

§. C^*-2-cat → connection.

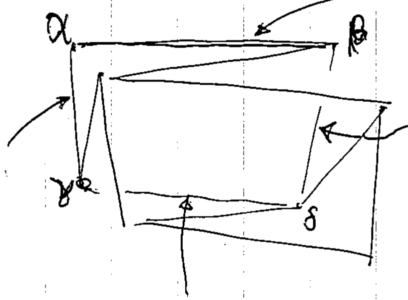
$$\mathcal{C}^\rho = \begin{pmatrix} \mathcal{C}^\rho_{00} & \mathcal{C}^\rho_{01} \\ \mathcal{C}^\rho_{10} & \mathcal{C}^\rho_{11} \end{pmatrix}$$

$\rho \in \mathcal{C}^\rho_{10}$ generates $\mathcal{C}^\rho$.

Recall

$$\begin{tikzcd}
\mathrm{Irr}\,\mathcal{C}^\rho_{00} \arrow[r, bend left, "\cdot\otimes\bar\rho"] \arrow[d, bend right, "\rho\otimes\cdot"'] & \mathrm{Irr}\,\mathcal{C}^\rho_{01} \arrow[l, bend left, "\cdot\otimes\rho"] \arrow[d, bend right, "\rho\otimes\cdot"'] \\
\mathrm{Irr}\,\mathcal{C}^\rho_{10} \arrow[u, bend right, "\bar\rho\otimes"'] \arrow[r, bend left, "\cdot\otimes\bar\rho"] & \mathrm{Irr}\,\mathcal{C}^\rho_{11} \arrow[l, bend left, "\cdot\otimes\rho"] \arrow[u, bend right, "\bar\rho\cdot\mathrm{id}"']
\end{tikzcd}$$

{ Graph



$\dim(\beta, \alpha\bar\rho)$ edges

$\dim(\gamma, \rho\alpha)$ edges

$\dim(\delta, \rho\beta)$ edges.

$\dim(\delta, \gamma\bar\rho)$ edges

We ~~want~~ add data of Morphisms of $\mathcal{C}^\rho$

We consider the oriented paths. fix. bijection.

upper horizontal.

$$\begin{tikzcd}
E(\alpha,\beta) \arrow[r, "T"] \arrow[d, "\sim"'] & \mathrm{ONB}(\beta, \alpha\bar\rho) \arrow[d, "r"] \\
E(\beta,\alpha) \arrow[r, "T"] & \mathrm{ONB}(\alpha, \beta\rho)
\end{tikzcd}$$

α: source vertex $\mathrm{Irr}\,\mathcal{C}_{00}$; ~~edo~~ β: range vertex $\mathrm{Irr}\,\mathcal{C}_{01}$; ↻; r: Frobenius map.

$$\xi \in E(\beta,\alpha) \longmapsto \sqrt{\frac{d(\rho)d(\alpha)}{d(\beta)}}\, T(\xi)^*\alpha(R_\rho) \in \mathrm{ONB}(\alpha, \beta\rho)$$

bottom. hor

$$\begin{tikzcd}
E(\delta,\gamma) \arrow[r, "T"] \arrow[d, "\sim"'] & \mathrm{ONB}(\gamma, \delta\rho) \arrow[d, "r"] & T(\xi) \arrow[d, mapsto, "\text{Frob. map}"] \\
E(\gamma,\delta) \arrow[r, "T"] & (\delta, \gamma\bar\rho) & \sqrt{\frac{d(\delta)d(\rho)}{d(\gamma)}}\, T(\xi)^*\delta(\bar R_\rho)
\end{tikzcd}$$

δ: $\mathrm{Irr}\,\mathcal{C}_{11}$ even; γ: $\mathrm{Irr}\,\mathcal{C}_{10}$ odd; ↻

left vertical

$$\begin{tikzcd}
E(\alpha,\gamma) \arrow[r, "S"] \arrow[d, "\sim"'] & (\gamma, \rho\alpha) \arrow[d, "\ell"] & S(\xi) \arrow[d, mapsto] \\
E(\gamma,\alpha) \arrow[r, "S"] & (\alpha, \bar\rho\gamma) & \sqrt{\frac{d(\rho)d(\gamma)}{d(\alpha)}}\, \bar\rho(S(\xi)^*) R_\rho
\end{tikzcd}$$

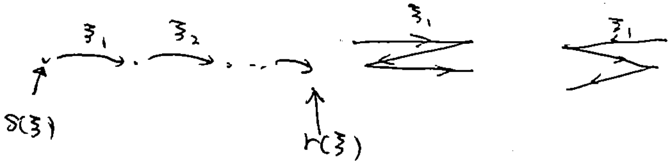
α: even $\mathcal{C}_{00}$; γ: odd $\mathcal{C}_{10}$

vertical right

$$\begin{tikzcd}
\underset{e_{11}\quad e_{01}}{E(\delta,\beta)} \arrow[r, "S"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & ONB(\beta,\bar{\rho}\delta) \arrow[d, "\varphi"] & \ni S(\xi) \arrow[d, mapsto] \\
E(\beta,\delta) \arrow[r, "S"] & ONB(\delta,\rho\beta) & \ni \sqrt{\dfrac{d(\rho)d(\delta)}{d(\beta)}}\,\rho(S(\xi)^{*})\bar{R}_{\rho}
\end{tikzcd}$$

For a long path (horizontal)

$\xi = \xi_1\,\xi_2 \cdots\cdots \xi_n$



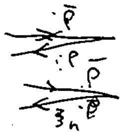
We set $T(\xi) = T(\xi_1)\cdots\cdots T(\xi_n)$

$\in (r(\xi), s(\xi)\bigcirc)$

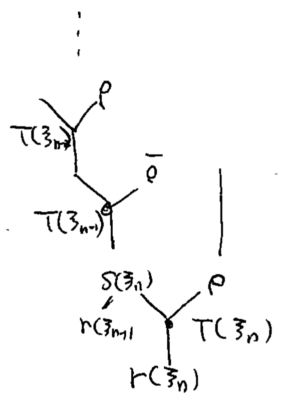
alternating words of ρ & $\bar{\rho}$

Eg.

$s(\xi) \in \mathrm{Irr}\,\mathcal{C}_{00}$ $\quad \underset{2m}{\overset{\shortparallel}{n}}$ even

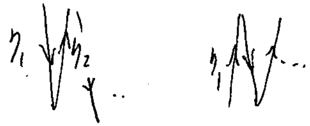


$T(\xi) \in (r(\xi), s(\xi)(\bar{\rho}\rho)^{n})$



For a long path (vertical)

$\eta = \eta_1 \cdots\cdots \eta_n$



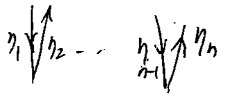
$S(\eta) \in (r(\eta),\ \cancel{S(\eta)}\bigcirc\ s(\eta))$

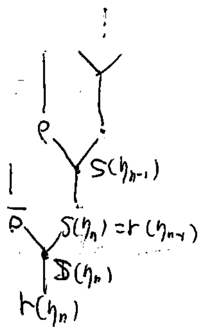
alternating words.

$S(\eta) = \bigcirc(S(\eta_1))\bigcirc(S(\eta_2)) \cdots \bigcirc(S(\eta_{n-1}))\,S(\eta_n)$

Eg.

$s(\xi) \in \mathrm{Irr}\,\mathcal{C}_{00}$ $\quad \underset{2m}{\overset{\shortparallel}{n}} = \text{even}$





$S(\eta) = \rho(\bar{\rho}\rho)^{m-1}(S(\eta_1))$

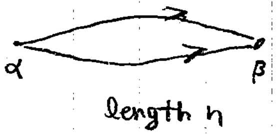
⋮

$\bar{\rho}\rho(S(\eta_{n-2}))$

$\rho(S(\eta_{n-1}))$

$S(\eta_n)$

Easy Fact



ONB$(\beta, \alpha \cdots)$ ↑ alternating words $\rho \cdot \bar{\rho}$

$\{T(\xi)\}_{\xi}$

vertical についても同様

Now. consider

$$\begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\zeta"'] & \beta \arrow[d, "\nu"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd}$$

α. δ を fix して

$\{ S(\zeta) T(\eta) \}^{(\gamma, \zeta, \eta)}$ ~~$\gamma\ S(\zeta) = \alpha$~~

$\{ (\rho \bar{\rho}) (T(\xi)) S(\nu) \} \ (\beta. \xi. \nu)$

ν の length

は $(\delta,\ \rho \cdot \bar{\rho} \cdots \cdots)$ の base を与える.

ここで.

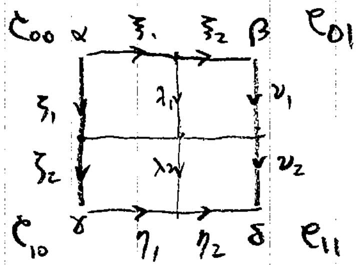
$$(\rho \cdot \bar{\rho})(T(\xi)) S(\nu) = \sum_{(\gamma, \zeta, \eta)} \begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\zeta"'] & \beta \arrow[d, "\nu"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} S(\zeta) T(\eta)$$

とおく (とくに ξ, ν ζ, η の長さが 1 のとき connection (cell?))

~~とおく S(ξ)~~

large conn. の値は length 1 の場合から計算できる.

Ex.



$$S(\zeta) T(\eta) = \bar{\rho}(S(\zeta_1)) S(\zeta_2) T(\eta_1) T(\eta_2)$$

$$\bar{\rho}\rho (T(\xi)) S(\nu) = \bar{\rho}\rho (T(\xi_1)) T(\xi_2) \bar{\rho}(S(\nu_1)) S(\nu_2)$$

$$= T(\eta_2)^* T(\eta_1)^* S(\zeta_2)^* \underset{\uparrow}{\ } \bar{\rho}(S(\zeta_1)^*) \rho(T(\xi_1)) \underset{\uparrow}{\ } T(\xi_2) \bar{\rho}(S(\nu_1)) S(\nu_2)$$

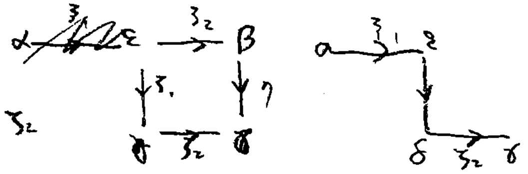
$\sum S(\lambda_2) S(\lambda_2)^*$ $\qquad$ $\sum_{\varepsilon,\ \cancel{x}} S(\lambda_1) S(\lambda_1)^*$

rectangle 以外.

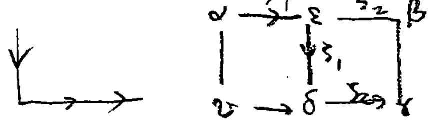
$$\begin{tikzcd}\alpha \arrow[r, "\xi_1"] & \varepsilon \arrow[r, "\xi_2"] & \beta \arrow[d, "\eta"] \\ & & \gamma\end{tikzcd}$$

= intertwiner のことを表すとする

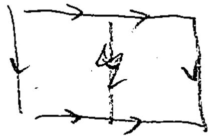
$= \sum_{\gamma, \xi_1, \xi_2}$



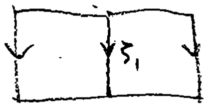
$= \sum$



よって



$= \sum_{\xi_1}$



となる. このように large conn は サイズ1のものから

計算できる.

Flat conn.

~~Std~~ lattice

$$\begin{tikzcd}
(1,1) \arrow[r] \arrow[d, "\rho"'] & (\bar\rho,\bar\rho) \arrow[r] \arrow[d, "\rho"'] & (\bar\rho\rho,\bar\rho\rho) \arrow[r] \arrow[d] & \boxed{(\bar\rho\rho\bar\rho,\bar\rho\rho\bar\rho)} \arrow[d, "|" marking] \\
\cancel{(1_M,1_M)\to}(\rho,\rho) \arrow[r] \arrow[d, "\bar\rho"'] & (\rho\bar\rho,\rho\bar\rho) \arrow[r] \arrow[d, "\bar\rho"'] & (\rho\bar\rho\rho,\rho\bar\rho\rho) \arrow[r] \arrow[d] & (\quad) \arrow[d] \\
\boxed{(\bar\rho\rho,\bar\rho\rho)} \arrow[r] & (\bar\rho\rho\bar\rho,\bar\rho\rho\bar\rho) \arrow[r] & ((\bar\rho\rho)^2,(\bar\rho\rho)^2) \arrow[r, "|" marking] & ((\bar\rho\rho)^2\bar\rho,(\bar\rho\rho)^2\bar\rho)
\end{tikzcd}$$

std. ness は たとえば ○ をつけたものが

ここで comm. することをいっていた.

これで conn の言葉にいいかえられる.

$$(\bar\rho\rho,\bar\rho\rho) = \sum_{\alpha<\bar\rho\rho} \mathrm{Span}(\alpha,\bar\rho\rho)(\alpha,\bar\rho\rho)^* \qquad S(\zeta)S(\nu)^*.$$

これの base は

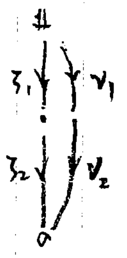
vertical path ζ $|\xi| = 2$

によって

$\{S(\xi)\}_{|\xi|=2}$ で与えられる.

$s(\zeta) = *$

$r(\xi) = \alpha$



同様に $(\bar\rho\rho\bar\rho,\bar\rho\rho\bar\rho)$ は

$$\{T(\xi)T(\eta)^*\}_{\beta<\bar\rho\rho\bar\rho}$$

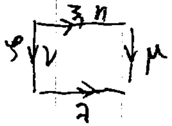
$s(\xi) = *,\ r(\xi) = \beta = r(\eta)$

であることがわかるから

Comm.

$$\Longleftrightarrow\ S(\zeta)S(\nu)^*\rho\bar\rho(T(\xi)T(\eta)^*) = \rho\bar\rho(T(\xi)T(\eta)^*)S(\zeta)S(\nu)^*$$

$\forall\alpha.\ \beta.\ \xi\ \eta.\ \zeta.\ \nu$



$$\Longrightarrow\ \cancel{S(\nu)^*\bar\rho\rho(T(\xi)) = S(\zeta)^*\bar\rho\rho(T(\xi)T(\eta)^*)S(\zeta)S(\nu)^*}$$

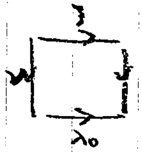
$\rho\bar\rho(T(\eta))$

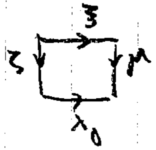
$$\sum_{\lambda\mu} S(\zeta)T(\lambda)\underbrace{T(\lambda)^*S(\nu)^*\ \rho\bar\rho(T(\xi))S(\mu)}S(\mu)^*\rho\bar\rho(T(\eta)^*)$$

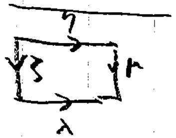
$$= \sum_{\lambda\mu} \rho\bar\rho(T(\xi))S(\mu)\underline{S(\mu)^*\rho\bar\rho(T(\eta)^*)}$$

$$\underline{S(\zeta)T(\lambda)}T(\lambda)^*S(\nu)^*$$

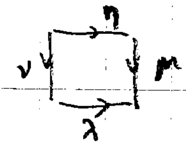
$T(\lambda)^*S(\zeta)^*$

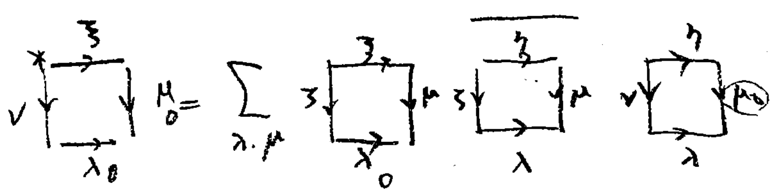
$\Longrightarrow \sum_{\mu}$  $S(\mu)^*\rho\bar\rho(T(\eta)^*)$

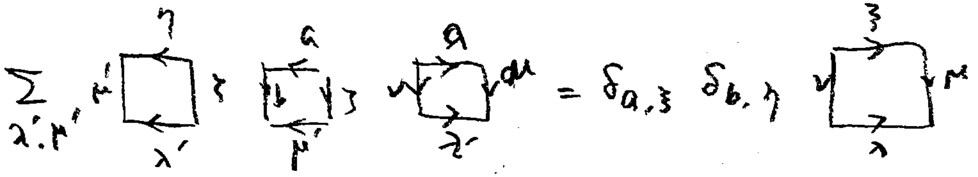
$= \sum_{\lambda\mu}$ 

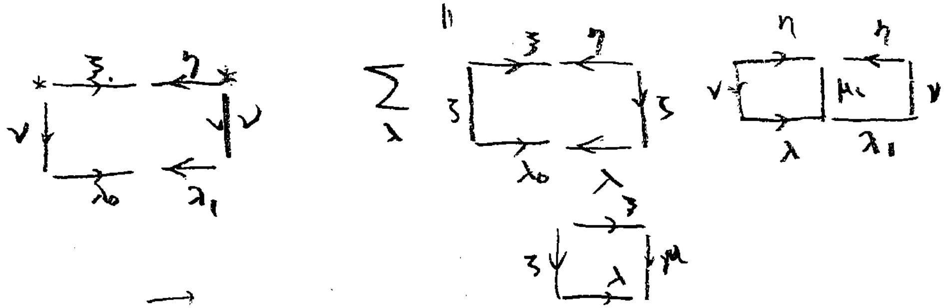


$T(\lambda)^*S(\nu)^*$







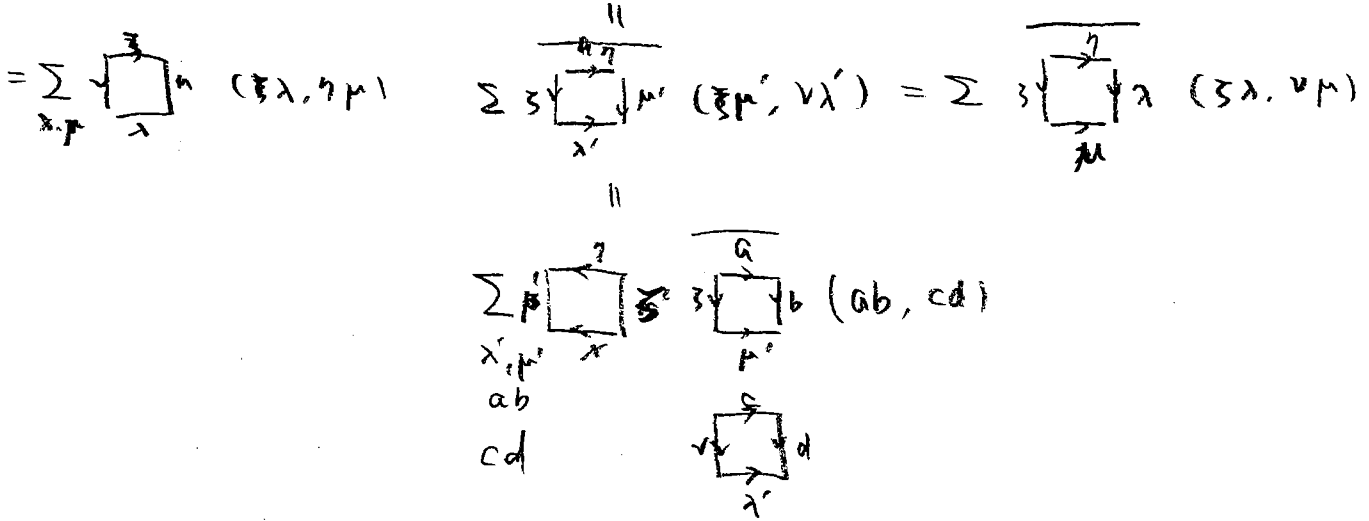


→

$\cancel{(\xi,\eta)}$

↓ $(\zeta,\nu)(\xi,\eta) = (\xi,\eta)(\zeta,\nu)$

$\sum(\zeta\lambda,\nu\lambda)(\xi\mu,\eta\mu) = \sum(\xi\mu',\eta\mu')(\zeta\lambda',\nu\lambda')$



§ Paragrps

C* 2cat $\mathcal{C}^{\rho}$ ⇝ connection $\begin{tikzcd}[sep=small] \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ & $\mu(\cdot)$ some function

length 1

To recover $\mathcal{C}^{\rho}$ ⇜ which axiom should be satisfied?

μは使えない

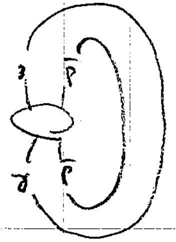
① unitarity

$$\sum_{\beta,\xi,\mu} \begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\lambda"'] & \beta \arrow[d, "\mu"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} \begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\lambda'"'] & \beta \arrow[d, "\mu"] \\ \gamma' \arrow[r, "\eta'"'] & \delta \end{tikzcd} = \delta_{\gamma,\gamma'}\delta_{\lambda\lambda'}\delta_{\eta,\eta'}$$

fix

$$\sum_{\gamma,\lambda,\eta} \begin{tikzcd} \alpha \arrow[r, "\xi'"] \arrow[d, "\lambda"'] & \beta' \arrow[d, "\mu'"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} \begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\lambda"'] & \beta \arrow[d, "\mu"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} = \delta_{\beta^{\circ},\beta^{\circ}}\delta_{\xi,\xi'}\delta_{\mu,\mu'}$$

(∵) Because of Base change matrix ∎



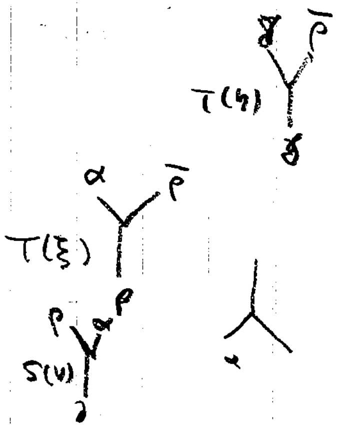
② Renormalization rule (Frob reciprocity)

$$\begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\nu"'] & \beta \arrow[d, "\zeta"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} = \sqrt{\frac{d(\nu)d(\delta)}{d(\beta)d(\gamma)}} \begin{tikzcd} \beta \arrow[r, "\tilde{\xi}"] \arrow[d, "\zeta"'] & \alpha \arrow[d, "\nu"] \\ \delta \arrow[r, "\tilde{\eta}"'] & \gamma \end{tikzcd}$$

$$\begin{tikzcd} \alpha \arrow[r, "\xi"] \arrow[d, "\nu"'] & \beta \arrow[d, "\zeta"] \\ \gamma \arrow[r, "\eta"'] & \delta \end{tikzcd} = \sqrt{\frac{d(\beta)d(\gamma)}{d(\alpha)d(\delta)}} \begin{tikzcd} \gamma \arrow[r, "\eta"] \arrow[d, "\tilde{\nu}"'] & \delta \arrow[d, "\tilde{\zeta}"] \\ \alpha \arrow[r, "\xi"'] & \beta \end{tikzcd}$$

(∵) we check

$$\begin{tikzcd} V_{00} \arrow[d, phantom, "\ni" rotate=90] & V_{01} \arrow[d, phantom, "\ni" rotate=90] \\ \alpha \arrow[r, "T(\xi)"] \arrow[d] & \beta \arrow[d, "T(\zeta)"] \\ \gamma \arrow[r] & \delta \\ V_{10} \arrow[u, phantom, "\in" rotate=90] & V_{11} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$



$$\begin{tikzcd} \beta \arrow[r, "\tilde{\xi}"] \arrow[d, "\zeta"'] & \alpha \arrow[d, "\nu"] \\ \delta \arrow[r, "\tilde{\eta}"'] & \gamma \end{tikzcd} = T(\tilde{\eta})^{*} S(\zeta)^{*} \rho(T(\tilde{\xi})) S(\nu)$$

$$= T(\tilde{\eta})^{*} S(\zeta)^{*} \rho\left(\sqrt{\frac{d(\alpha)d(\rho)}{d(\beta)}}\, T(\xi)^{*}\alpha(R_{\rho})\right) S(\nu)$$

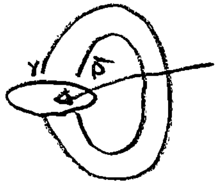
$$= \sqrt{\frac{d(\alpha)d(\rho)}{d(\beta)}}\, S(\nu)^{*}\rho(\alpha(R_{\rho}^{*}))\,\rho(T(\xi))\,S(\zeta)\,T(\tilde{\eta})$$

$$= \sqrt{\frac{d(\alpha)d(\rho)}{d(\beta)}}\,\gamma(R_{\rho}^{*})\,\underline{S(\nu)^{*}\rho(T(\xi))\,S(\zeta)}\,\cancel{T(\tilde{\eta})}$$

$$\sqrt{\frac{d(\gamma)d(\rho)}{d(\delta)}}\,\underline{T(\eta)^{*}}\,\gamma(R_{\rho})$$

$$\gamma\bar{\rho} \leftarrow \rho\alpha\bar{\rho} \leftarrow \rho\beta \leftarrow \delta \longleftarrow \gamma\bar{\rho}$$

$= \sqrt{\frac{d(\alpha)d(\gamma)}{d(\beta)d(\delta)}}\, d(\rho)$



$S(\nu)^* \rho(T(\xi))\, S(\zeta)\, T(\eta)^*$

$= \sqrt{\frac{d(\alpha)d(\gamma)}{d(\beta)d(\delta)}}\, d(\rho) \cdot \frac{1}{d(\gamma)d(\rho)}\, \mathrm{Tr}_{\gamma\alpha\bar\rho}($

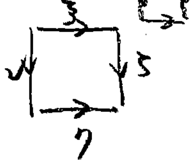
$)$

$= \sqrt{\frac{d(\alpha)d(\gamma)}{d(\beta)d(\delta)}}\, \frac{1}{d(\gamma)}\, \mathrm{Tr}_\delta\left(\boxed{T(\eta)^* S(\nu)^* \rho(T(\xi))\, S(\zeta)}\right)$

$\in (\delta,\delta) = \mathbb{C}1$

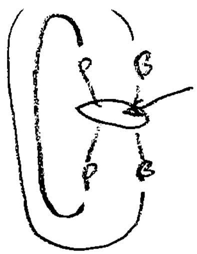
$d(\delta)$

$= \sqrt{\frac{d(\alpha)d(\delta)}{d(\beta)d(\gamma)}}$



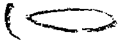
⌟

$= d(\rho)\sqrt{\frac{d(\alpha)d(\beta)}{d(\gamma)d(\delta)}}$



$\boxed{S(\zeta)}\, T(\eta)^* S(\nu)^* \rho(T(\xi))$

$= d(\rho)\sqrt{\frac{d(\alpha)d(\beta)}{d(\gamma)d(\delta)}}\, \frac{1}{d(\rho)d(\beta)}\, \mathrm{Tr}_{\rho\beta}($

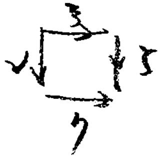


$)$

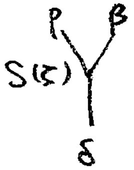
$= \sqrt{\frac{d(\alpha)d(\beta)}{d(\gamma)d(\delta)}}\, \frac{1}{d(\rho)}\, \mathrm{Tr}_\delta\left(T(\eta)^* S(\nu)^* \rho(T(\xi))\, S(\zeta)\right)$

$\in (\delta,\delta) = \mathbb{C}$

$= \sqrt{\frac{d(\alpha)d(\delta)}{d(\beta)d(\gamma)}}$

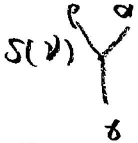


$$\begin{tikzcd}
\gamma \arrow[r, "\eta"] \arrow[d, "\tilde\nu"'] & \delta \arrow[d, "\tilde\zeta"] \\
\alpha \arrow[r, "\xi"'] & \beta
\end{tikzcd} = \overline{T(\xi)^* S(\tilde\nu)^* \bar\rho(T(\eta)) S(\tilde\zeta)}$$



$= \cancel{T(\xi)^*}$

$S(\tilde\zeta)^* \bar\rho(T(\eta)^*)\, S(\tilde\nu)\, T(\xi)$



$= \sqrt{\frac{d(\rho)d(\beta)}{d(\delta)}}\left(\bar\rho(S(\zeta)^*) R_\rho\right)^* \bar\rho(T(\eta)^*)\, \sqrt{\frac{d(\rho)d(\alpha)}{d(\gamma)}}\, \bar\rho(S(\nu)^*)\, R_\rho T(\xi)$

$= d(\rho)\sqrt{\frac{d(\beta)d(\alpha)}{d(\delta)d(\gamma)}}\, R_\rho\left(\bar\rho\left[S(\zeta)T(\eta)^* S(\nu)^* \rho(T(\xi))\right]\right) R_\rho$

$\rho\beta \leftarrow \delta \leftarrow \gamma\bar\rho \longleftarrow \rho\alpha\bar\rho \leftarrow \rho\beta$

unitarity と renormalization をみたす

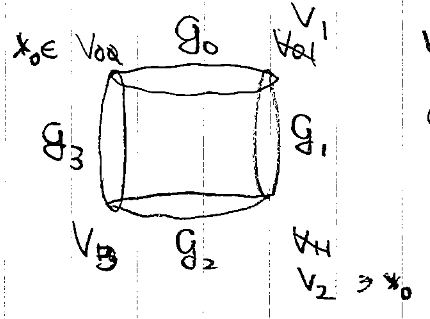
connection がそろったとき.

lattice をつくることができること

$$\begin{tikzcd}
 & \text{graph with } W(\cdot) \arrow[dr] & \\
\mathcal{C}^P \arrow[ur, "\text{(Fusion graph \& 6j symbol) done}"] \arrow[rr, "\text{done}"'] & & \text{lattice}
\end{tikzcd}$$

Given data.

- 4 graphs



vertex 共有.

connected

- Dimensions

$\mu : V_0 \sqcup V_1 \sqcup V_2 \sqcup V_3 \longrightarrow [1,\infty)$

$\beta \geq 1$

$\mu(*_0) = 1 = \mu(*_1)$

s.t.

$\Lambda \vec{\mu}_\circ = \beta \vec{\mu}_\square$

↑

adjacency matrix $\Lambda_{X,Y} = \#$ edges $X - Y$.

- Connection W

$$W\left(\begin{tikzcd}
\alpha \arrow[r, "\xi"] \arrow[d, "\zeta"'] & \beta \arrow[d, "\nu"] \\
\gamma \arrow[r, "\eta"'] & \delta
\end{tikzcd}\right) \in \mathbb{C}$$

適宜省略. length 1.

$\xi, \eta \in \text{edge}\,\mathcal{G}_0 \cup \text{edge}\,\mathcal{G}_2$

$\zeta, \nu \in \text{edge}\,\mathcal{G}_1 \cup \text{edge}\,\mathcal{G}_3$

- Unitarity
- Renormalization rule.

STEP1

String algs (horizontal)

$H^n :=$

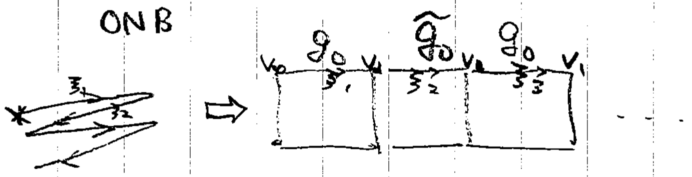
$H^n_\alpha := \text{span}\{\xi \mid \xi = \xi_1 \cdots \xi_n$ length n path in $\mathcal{G}_0$

formal

$s(\xi_1) = *$.

$r(\xi_1) = s(\xi_2) \cdots \quad r(\xi_n) = \alpha\}$

ONB



$$B(H^n_\alpha) := \mathrm{span}\{(\xi,\eta) \mid *_0 \overset{\xi}{\underset{\eta}{\rightrightarrows}} \alpha \ \text{length } n \ \& \ \xi,\eta \text{ in } \mathcal{G}_0\}$$

matrix units
w.r.t. ONB $\{\xi\}$

$$\mathrm{String}^{(n)}\mathcal{G}_0 = \mathrm{span}\{(\xi,\eta) \mid *_0 \overset{\xi}{\underset{\eta}{\rightrightarrows}} \bullet \}$$

closed string

common vertex

$r(\xi) = r(\eta)$

$$= \bigoplus_{\alpha \in V_n} B(H^n_\alpha)$$

mod 2

$$1 = \sum_\xi (\xi,\xi)$$

$$\begin{tikzcd} \mathrm{String}^{(n)}\mathcal{G}_0 \arrow[r] & \mathrm{String}^{(n+1)}\mathcal{G}_0 \\ (\xi,\eta) \arrow[r, mapsto] & \sum (\xi\zeta, \eta\zeta) \end{tikzcd}$$

$\zeta : s(\zeta) = r(\xi) = r(\eta)$

$|\zeta| = 1$

unital
$*$-homo.
faithful

$$\begin{matrix} \mathbb{C} \\ \| \\ \mathrm{String}^{(0)}\mathcal{G}_0 \end{matrix} \longrightarrow \mathrm{String}^{(1)}\mathcal{G}_0 \longrightarrow \mathrm{String}^{(2)}\mathcal{G}_0 \longrightarrow \cdots$$

Similarly we have

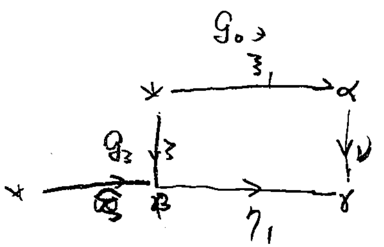
$$\mathrm{String}^{(0)}\mathcal{G}_1 \longrightarrow \mathrm{String}^{(1)}\mathcal{G}_1 \longrightarrow \mathrm{String}^{(2)}\mathcal{G}_1 \longrightarrow \cdots$$

STEP2. ~~The~~ Lattice C^*-algs

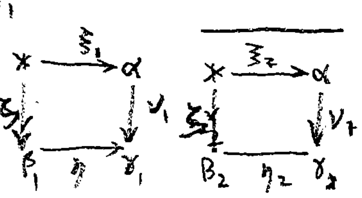
We construct the following vertical homos

$$\begin{tikzcd} & \mathrm{String}^{(0)}\mathcal{G}_0 \arrow[r] \arrow[d, dashed] & \mathrm{String}^{(1)}\mathcal{G}_0 \arrow[r] \arrow[d, dashed] & \cdots \arrow[r] & \mathrm{String}^{(n)}\mathcal{G}_0 \arrow[r] \arrow[d, dashed] & \mathrm{String}^{(n+1)}\mathcal{G}_0 \arrow[r] \arrow[d, dashed] & \cdots \\ \mathrm{String}^{(0)}\mathcal{G}_1 \arrow[r] & \mathrm{String}^{(1)}\mathcal{G}_1 \arrow[r] & \mathrm{String}^{(2)}\mathcal{G}_1 \arrow[r] & \cdots & \mathrm{String}^{(n+1)}\mathcal{G}_1 \arrow[r] & \mathrm{String}^{(n+2)}\mathcal{G}_1 \arrow[r] & \cdots \end{tikzcd}$$

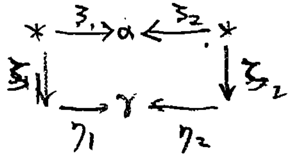
$$\begin{tikzcd} \mathrm{String}^{(n)}\mathcal{G}_0 \ni (\xi_1, \xi_2) \arrow[d, "\varphi_n"] \\ \mathrm{String}^{(n+1)}\mathcal{G}_1 \end{tikzcd}$$



$$\sum_{\substack{\beta_1\beta_2\gamma \\ \xi_1,\eta_1,\nu_1 \\ \omega_1,\omega_2 \\ \xi_2\eta_2\nu_2}}$$



$$(\hat{\xi}_1 \eta_1, \hat{\xi}_2 \eta_2)$$



$$(\xi_1, \xi_2)\cdot(\xi_3, \xi_4) = \delta_{\xi_2, \xi_3}\ (\xi_1, \xi_4)$$

$$\downarrow$$

$$\sum \begin{tikzcd} {} \arrow[r, "\xi_1"] \arrow[d, "\zeta_1"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta_1"'] & {} \end{tikzcd} \ \overline{\begin{tikzcd} {} \arrow[r, "\xi_2"] \arrow[d, "\zeta_2"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta_2"'] & {} \end{tikzcd}} \ \underbrace{\begin{tikzcd} {} \arrow[r, "\xi_3"] \arrow[d, "\zeta_2'"'] & {} \arrow[d, "\nu'"] \\ {} \arrow[r, "\eta_2'"'] & {} \end{tikzcd}}_{\delta_{\xi_2, \xi_3}\ \delta_{\nu, \nu'}} \ \overline{\begin{tikzcd} {} \arrow[r, "\xi_4"] \arrow[d, "\zeta_3"'] & {} \arrow[d, "\nu'"] \\ {} \arrow[r, "\eta_3"'] & {} \end{tikzcd}} \ (\hat{\zeta}_1\eta_1, \hat{\zeta}_3\eta_3)$$

k_n $*$-homo

diagram comm.

~~$(\xi_1, \xi_2) \mapsto$~~

$$\begin{tikzcd} H^n \arrow[d, "V_n(\nu)"'] & \xi \arrow[d, mapsto] & & \sum\limits_{\xi} \overline{\begin{tikzcd} {} \arrow[r, "\xi"] \arrow[d, "\zeta"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta"'] & {} \end{tikzcd}}\ \xi \\ H^{n+1} & \sum\limits_{\zeta, \eta} \begin{tikzcd} {} \arrow[r, "\xi"] \arrow[d, "\zeta"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta"'] & {} \end{tikzcd}\ \hat{\zeta}\eta & & \hat{\zeta}\eta \arrow[u, mapsto] \end{tikzcd}$$

isometry

$$\sum_{\nu} V_n(\nu)V_n(\nu)^* = 1.$$

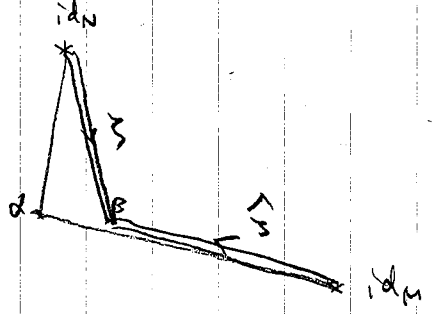
$$k_n(x) = \sum_{\nu} V_n(\nu)\, x\, V_n(\nu)^*$$

$$\begin{tikzcd} ((\bar{\rho}\rho)^n, (\bar{\rho}\rho)^n) \arrow[d, "\rho\circ"] & T(\xi_1)T(\xi_2)^* \arrow[d, mapsto] \\ (\rho(\bar{\rho}\rho)^n, \rho(\bar{\rho}\rho)^n) & \rho(T(\xi_1)T(\xi_2)^*) \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[r, "\xi"] \arrow[d, "\zeta"'] & \cdot \arrow[d, "\nu"] \\ {} \arrow[r, "\eta_1"'] & \gamma \end{tikzcd}$$

$$= \sum \rho(T(\xi_1))S(\nu)\ S(\nu)^*\rho(T(\xi_2))^*$$

$$= \sum \begin{tikzcd} {} \arrow[r, "\xi_1"] \arrow[d, "\zeta_1"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta_1"'] & {} \end{tikzcd} \ \overline{\begin{tikzcd} {} \arrow[r, "\xi_2"] \arrow[d, "\zeta_2"'] & {} \arrow[d, "\nu"] \\ {} \arrow[r, "\eta_2"'] & {} \end{tikzcd}} \ S(\zeta_1)T(\eta_1)T(\eta_2)^*S(\zeta_2)^*$$



$$\begin{tikzcd} (\xi_1, \xi_2) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & \sum (\xi_1\lambda, \xi_2\lambda) \arrow[d, mapsto] \\ \sum V_n(\nu)(\xi_1, \xi_2)V_n(\nu)^* & \sum\limits_{\lambda, \nu_2} V_{n+1}(\nu_2)\ (\xi_1\lambda, \xi_2\lambda)\, V_{n+1}(\nu_2)^* \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\xi_1\lambda"] \arrow[d, "\zeta"'] & {} \arrow[d, "\nu_2"] \\ {} \arrow[r, "\eta\ \omega"'] & {} \end{tikzcd} \ \hat{\zeta}\eta\omega$$

$$V_n(\nu_1)\xi_1 = \sum \begin{tikzcd} {} \arrow[r, "\xi_1"] \arrow[d, "\zeta"'] & {} \arrow[d, "\nu_1"] \\ {} \arrow[r, "\eta"'] & {} \end{tikzcd} \ \hat{\zeta}\eta \mapsto \sum \begin{tikzcd} {} \arrow[r, "\xi_1"] \arrow[d, "\zeta"'] & {} \arrow[d, "\nu_1"] \\ {} \arrow[r, "\eta"'] & {} \end{tikzcd} \ \hat{\zeta}\eta\omega$$

biunitarity

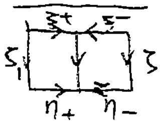
STEP3.

left Inverses

$$\begin{tikzcd}
\mathrm{String}^{n}\mathcal{G}_0 \arrow[r, "i_n"] \arrow[d, "k_n"'] & \mathrm{String}^{n+1}\mathcal{G}_0 \arrow[d, "k_{n+1}"] \\
\mathrm{String}^{n+1}\mathcal{G}_1 \arrow[r, "j_n"] & \mathrm{String}^{n+2}\mathcal{G}_1
\end{tikzcd}$$

$$\phi_{i_n}((\xi^+_+, \xi^-_-)) := \delta_{\xi^+_{n+1}, \xi^-_{n+1}} \frac{\mu(r(\xi))}{\beta\mu(r(\xi^+_n))} (\xi^+_1 \cdots \xi^+_n, \xi^-_1 \cdots \xi^-_n)$$

$\xi^{\pm} = \xi^{\pm}_1 \cdot \quad \xi^{\pm}_{n+1}$

$$\phi_{k_n}((\eta^+, \eta^-)) := \sum_{\xi^+, \xi^-} \frac{d(r(\eta_+))}{\beta\mu(r(\xi^+))}$$ 
(ξ^+, ξ^-)

$$\eta^{\pm} = \hat{\xi}^{\pm}\eta^{\pm}_1 \cdots \eta^{\pm}_n.$$

$$\phi_{i_n} \cdot i_n((\xi, \eta)) = \sum_{r(\xi)=\alpha} \phi_{i_n}((\xi\lambda, \eta\lambda)) \ \xi$$

$$= \sum_{\lambda} \frac{1}{\beta} \frac{\mu(\cancel{\alpha})}{\mu(\alpha)} (\xi, \eta)$$

$$\alpha \xrightarrow{\lambda} \cancel{\beta}\gamma$$

$$= \sum_{\gamma} \frac{h_{\alpha,\gamma}\,\mu(\gamma)}{\mu(\alpha)} \frac{1}{\beta} (\xi, \eta)$$

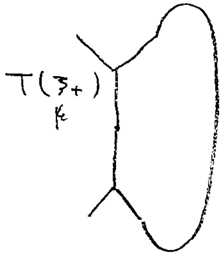
$$= (\xi, \eta).$$

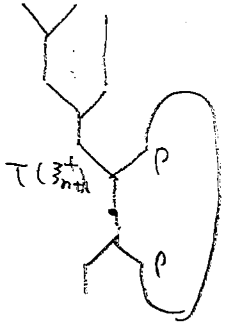
STEP4 tracial state.

$$\mathrm{tr}((\xi_+, \xi_-)) := \delta_{\xi_+, \xi_-} \beta^{-n} \mu(r(\xi_+))$$

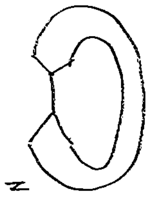
$$(\xi_+, \xi_-) \in \mathrm{String}^{(n)}\mathcal{G}_*$$

2周からくるとき

$$\phi_{i_n}(T(\xi_+)T(\xi_-)^*) =$$ 

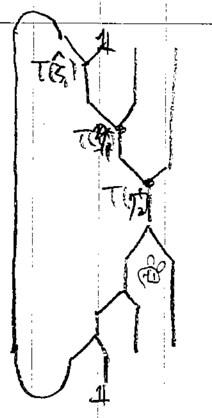
$$=$$ 

$$= T(\xi^+_1 \cdots \xi^+_n) T(\xi^-_1 \cdots \xi^-_n)^*$$

$$\delta_{r(\xi^+_n)\, r(\xi^-_n)}$$ 

$$\frac{d(r(\xi))}{d(r(\xi^+_n))d(\rho)} \delta_{\xi^+_{n+1}, \xi^-_{n+1}} =$$

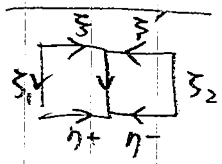
$\Phi_{\mathfrak{k}_n}((\hat{\xi}_1\eta_k^+, \hat{\xi}_2\eta_k^-)) =$



$$= \sum \delta_{r(\xi), r(\xi')} \frac{d(r(\eta_+))}{d(\rho)d(r(\xi))} \delta_{\nu,\nu'} \cancel{\beta} T(\xi)T(\xi')^*.$$

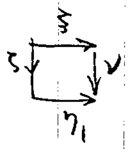
$$\cancel{\frac{d(r(\eta_-))}{d(\rho)d(\eta)}}$$

$$= \sum \delta_{r(\xi), r(\xi')} \frac{d(r(\eta_+))}{\beta\, d(r(\xi))}$$



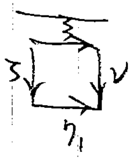
(ξ, ξ')

$T(\hat{\xi}) = S(\xi)$

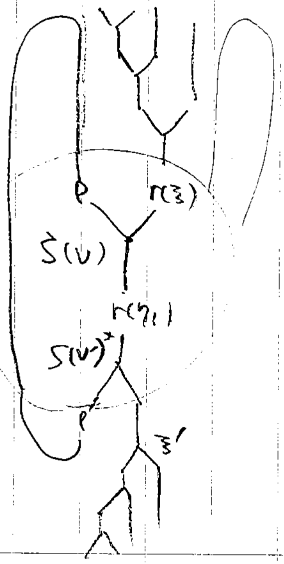


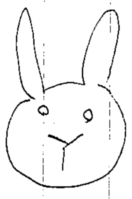
$T(\hat{\xi}, \eta_1^+) = S(\xi)T(\eta_1^+)$

$= \sum_{\xi,\nu}$



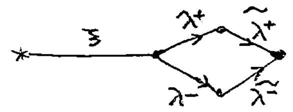
$\rho(T(\xi))\cancel{\beta}S(\nu))$



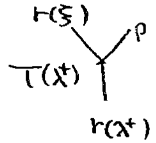


STEP5 Jones proj

2圏のとき



$$(\xi\lambda^+\tilde\lambda^+, \xi\lambda^-\tilde\lambda^-) = T(\xi)T(\lambda^+)T(\tilde\lambda^+)T(\tilde\lambda^-)^*T(\lambda^-)^*T(\xi)^*$$

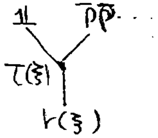


$$= T(\xi)T(\lambda^+)\cdot \sqrt{\frac{d(r(\xi))d(\rho)}{d(r(\lambda^+))}}\sqrt{\frac{d(r(\xi))d(\rho)}{d(r(\lambda^-))}}\; T(\lambda^+)^* \, r(\xi)(\bar R_\rho \bar R_\rho^*)\, T(\lambda^+)T(\lambda^-)^*T(\xi)^*$$

$$\frac{1}{d(\rho)}\sum_{\xi\lambda^+,\lambda^-}\frac{\sqrt{d(r(\lambda^+))d(r(\lambda^-))}}{d(r(\xi))}(\xi\lambda^+\tilde\lambda^+, \xi\lambda^-\tilde\lambda^-)$$

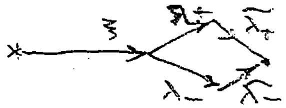
$$= \sum_{\xi\lambda^+\lambda^-} T(\xi)\, T(\lambda^+)\underset{\underset{r(\xi)(\bar R_\rho \bar R_\rho^*)}{}}{T(\lambda^+)^*\,T(\lambda^-)}\, T(\lambda^-)^*\, T(\xi)^*$$

$$= \bar\rho\rho\cdot\,(\bar R_\rho \bar R_\rho^*)$$



$$e_n := \frac{1}{\beta}\sum_{\xi,\lambda_+\lambda_-}\frac{\sqrt{\mu(r(\lambda_+))\,\mu(r(\lambda_-))}}{\mu(r(\xi))}(\xi\lambda_+\tilde\lambda_+, \xi\lambda_-\tilde\lambda_-)$$

$$\in \mathrm{String}^{(n)}\mathcal{G}_0$$



$$f_1 := \frac{1}{\beta}\sum_{\substack{\lambda_+\lambda_-\\ |\lambda_+|=1=|\lambda_-|}}\sqrt{\mu(r(\lambda_+))\,\mu(r(\lambda_-))}\,(\lambda_+\tilde\lambda_+, \lambda_-\tilde\lambda_-)$$

$$\in \mathrm{String}^{(2)}\mathcal{G}_1$$

→ STEP 1 ～ 5 より

biunitary conn. ⤳ β-lattice ができる.

$$\begin{tikzcd}
 & & \mathrm{Strg}^2\mathfrak{g}_0 \arrow[r] \arrow[d, "k_0"] & \mathrm{Str}^1\mathfrak{g}_0 \arrow[r] \arrow[d, "k_1"] & \cdots \\
 & \mathrm{Str}^0\mathfrak{g}_1 \arrow[r] \arrow[d, "l_0"] & \mathrm{Str}^1\mathfrak{g}_1 \arrow[r] \arrow[d, "l_1"] & \mathrm{Str}^2\mathfrak{g}_1 \arrow[r] \arrow[d, "l_2"] & \cdots \\
\mathrm{Str}^0\mathfrak{g}_0 \arrow[r] & \mathrm{Str}^1\mathfrak{g}_0 \arrow[r] & \mathrm{Str}^2\mathfrak{g}_0 \arrow[r] & \mathrm{Str}^3\mathfrak{g}_0 \arrow[r] & \cdots
\end{tikzcd}$$

l_n が 何かを記述する.

$$\Psi_1(x) = \lim_{r} W_n\, x\, W_n^*$$

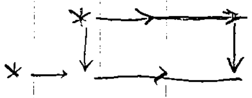
$$W_n = \frac{1}{\sqrt{\beta}^{\,n-1}} f_1 f_2 \cdots f_n$$

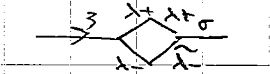
$$x = (\hat{\xi}_+ \overset{n-1}{\eta_+}, \hat{\xi}_- \overset{n-1}{\eta_-}) \in \mathrm{String}^n \mathfrak{g}_1$$

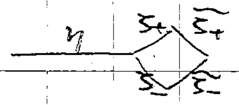
$$\Psi_1(x) = \frac{\cancel{1}\ \beta^{n-\cancel{1}}}{\beta^{\cancel{n-1}}} f_1 f_2 \cdots f_{n+2} (\hat{\xi}_+\eta_+, \hat{\xi}_-\eta_-) f_{n+2} \cdots f_1$$

$$\Big\downarrow\ \beta^{*a}\ \phi_{k_{n+2}}(\cdot)$$

$$l_n(x).$$







$$f_1 f_2 \equiv \frac{1}{\beta^2} \sum \frac{\sqrt{\mu(r(\lambda_+))\mu(r(\lambda_-))}}{\mu(r(\xi))} (\xi\lambda_+\widetilde{\lambda_+}\sigma, \xi\lambda_-\widetilde{\lambda_-}\textcircled{\sigma})$$

$$\frac{\sqrt{\mu(r(\xi_+))\,\mu(r(\xi_-))}}{\mu(r(\eta))} (\eta\xi_+\textcircled{\widetilde{\xi_+}}, \eta\xi_-\widetilde{\xi_-})$$

$\sigma = \widetilde{\xi_+}$
$\widetilde{\lambda_-} = \xi_+$

$$f_{n+2}(\hat{\xi}_+\eta_1^+ \cdots \eta_{n-1}^+, \hat{\xi}_-\eta_1^- \cdots \eta_{n-1}^-)$$

$$= \frac{1}{\beta} \sum_{\lambda_+,\lambda_-} \frac{\sqrt{\mu(r(\lambda_+))\mu(r(\lambda_-))}}{\mu(r(\eta_{n-1}^+))} (\underbrace{\hat{\xi}_+\eta_1^+ \cdots \eta_{n-1}^+\lambda_+}_{n+1}\widetilde{\lambda_+}, \hat{\xi}_-\eta_1^- \cdots \eta_{n-1}^-\lambda_-\widetilde{\lambda_-})$$

$$f_{n+1} f_{n+2}$$

$$= \frac{1}{\beta^2} \sum_{\substack{\lambda_+\lambda_- \\ \sigma_+\sigma_- \\ \nu}} \frac{\sqrt{\mu(r(\lambda_+))\mu(r(\lambda_-))}}{\mu(r(\eta_{n-1}^+))} \frac{\sqrt{\mu(r(\sigma_+))\ \mu(r(\sigma_-))}}{\mu(r(\eta_{n-2}^+))}$$

$$(\hat{\xi}_+\eta_1^+ \cdots \eta_{n-2}^+\sigma_+\widetilde{\sigma_+}\nu, \hat{\xi}_+\eta_1^+ \cdots \eta_{n-2}^+\sigma_-\widetilde{\sigma_-}\nu)(\qquad)$$

$\nu = \widetilde{\lambda_+} = \eta_{n-1}^+$
$\widetilde{\sigma_-} = \lambda_+ = \widetilde{\eta_{n-1}^+}$
$\sigma_- = \eta_{n-1}^+$

$$= \frac{1}{\beta^2} \sum \frac{\sqrt{\mu(r(\overset{\mu(r(\eta_{n-2}^+))}{\eta_{n-1}^+}))\,\mu(r(\lambda_-))}}{\mu(r(\eta_{n-1}^+))} \frac{\sqrt{\mu(r(\sigma_+))\,\mu(r(\eta_{n-1}^+))}}{\mu(r(\eta_{n-2}^+))}$$

$$(\hat{\xi}_+\eta_1^+ \cdots \eta_{n-2}^+\sigma_+\widetilde{\sigma_+}\eta_{n-1}^+, \hat{\xi}_-\eta_1^- \cdots \eta_{n-1}^-\lambda_-\widetilde{\lambda_-})$$

$$f_{n+1}f_{n+2}(\quad)f_{n+2}f_{n+1}$$

$$= \frac{1}{\beta^2} \sum \frac{\mu(r(\sigma_-))\sqrt{\mu(r(\sigma_+))\,\mu(r(\eta_{n-2}^+))\,\mu(r(\eta_{n-2}^-))}}{\mu(r(\eta_{n-2}^+))\mu(r(\eta_{n-2}^-))}$$

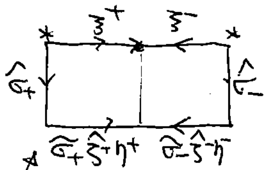
$$(\hat{\xi}_+\eta_1^+ \cdots \eta_{n-2}^+\sigma_+\widetilde{\sigma_+}\eta_{n-1}^+\ \hat{\xi}_-\eta_1^- \cdots \eta_{n-2}^-\sigma_-\widetilde{\sigma_-}\eta_{n-1}^-)$$

$\rightsquigarrow f_1 \cdots f_{n+2}(\hat{\xi}^+\eta^+, \hat{\xi}^-\eta^-) f_{n+2} \cdots f_1$

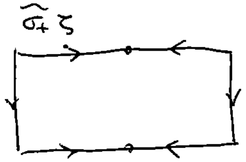
$= \frac{1}{\beta^{n+2}} \sum_{\sigma_+,\sigma_-} \sqrt{\mu(r(\sigma_+))\mu(r(\sigma_-))} \quad (\sigma_+\widetilde{\sigma_+}\hat{\xi}^+\eta^+, \sigma_-\widetilde{\sigma_-}\hat{\xi}^-\eta^-)$

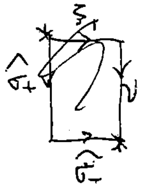
$\mapsto l_n((\hat{\xi}^+\eta^+, \hat{\xi}^-\eta^-))$

$= \sum_{\sigma_+\sigma_-} \sqrt{\mu(r(\sigma_+))\mu(r(\sigma_-))} \quad \frac{\mu(r(\eta^+))}{\beta\,\mu(r(\xi^+))} \quad (\xi^+, \xi^-)$

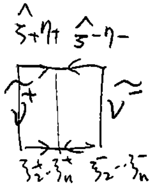


$\sum$

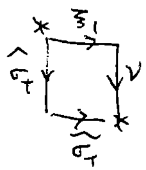




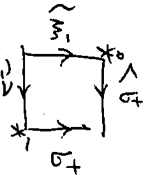
$\frac{1}{\beta} \sum A_{\nu_+\widetilde{\xi_1}} \, A_{\nu_-\widetilde{\xi_1^-}}$



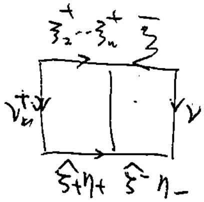
$\widetilde{\nu} \quad (\xi^+, \xi^-)$



$= \sqrt{\frac{\mu(r(\sigma_+))\mu(\xi_1)}{\mu(*)^2}}$



$= \sum \sqrt{\mu(r(\xi_1^+))\mu(r(\xi_1^-))} \quad A_{\nu_+\widetilde{\xi_1}} \, \overline{A_{\nu_-\widetilde{\xi_1^-}}}$

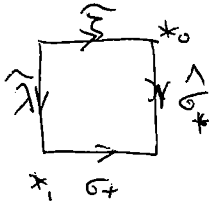


$\nu_- \quad (\xi^+, \xi^-)$

$\frac{\mu(r(\eta^+))}{\beta\,\mu(r(\xi^+))}$

⟨Initialization⟩

$\sum_{\sigma_+}$



$\mu(r(\sigma_+)) = \delta_{\xi,\hat{\lambda}} \cdot (\sqrt{\beta})$.

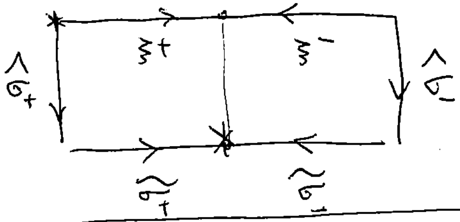
$$\begin{tikzcd} S_n{}^1 g_0 \arrow[d, "k_1"] \\ S_n{}^2 g_1 \end{tikzcd}$$

$\phi_{k_1}(f_1) = 1/\beta \xrightarrow[\text{~~tor.~~ Markov}]{\text{vertical}}$

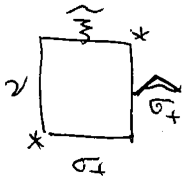
$\phi_{k_1}(f_1) = \sum_{\sigma_+\sigma_-} \frac{1}{\beta} \cdot \frac{\sqrt{\mu(r(\sigma_+))\mu(r(\sigma_-))}}{1} \quad \phi_{k_1}((\sigma_+\widetilde{\sigma_+}, \sigma_-\widetilde{\sigma_-}))$

$= \frac{1}{\beta} \sum_{\sigma_+\sigma_-} \sqrt{\mu(r(\sigma_+))\mu(r(\sigma_-))} \quad (\xi^+, \xi^-)$

$\frac{\cancel{\mu(r(\sigma_+))}}{\beta\,\mu(r(\xi^+))}$.



$A_{\nu,\widetilde{\xi}} = \sum_{\sigma_+}$



$\mu(r(\sigma_+))$. ~~unitary~~.

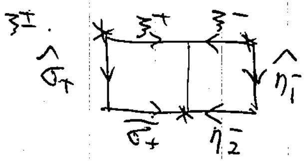
β

$A_{\nu,\xi_1}^* A = \beta \, 1$

$\phi_{R_1}(f_1(\eta_1^+\eta_2^+, \eta_1^-\eta_2^-))$

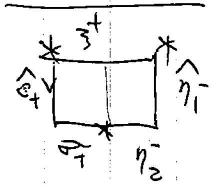
$= \delta_{\eta_1^+,\widehat{\eta_2^+}} \sum_{\sigma_+} \frac{1}{\beta} \sqrt{\mu(r(\sigma_+))\,\mu(r(\eta_1^+))} \cdot \phi_{R_1}((\sigma_+\widehat{\sigma_+}, \eta_1^-\eta_2^-))$

$= \cancel{\delta_{\eta_1^+\widehat{\eta_2^+}}} \frac{1}{\beta} \sum_{\sigma_+, \xi^\pm} \sqrt{\mu(r(\sigma_+))\,\mu(r(\eta_1^+))} \cdot \frac{1}{\beta} \cancel{\sqrt{\mu(r(\sigma_+))\,\mu(r(\eta_1^-))}} \cdot \frac{\mu(*_1)}{\mu(r(\xi^+))}$



(ξ^+, ξ^-)

$= \frac{1}{\beta^2} \mu(r(\eta_1^+)) \sum_{\sigma_+, \xi^+} \frac{\sqrt{\mu(r(\sigma_+))\,\cancel{\mu(r(\eta_1^+))}}}{\mu(r(\xi^+))}$

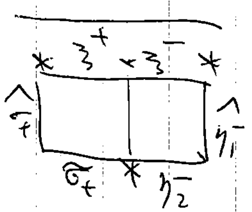


(ξ^+, ξ^-)

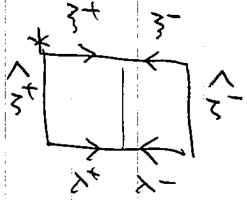
$\eta_2^+ = \widehat{\eta_1^+}$

$E_{R_1}(f_1(\quad))$

$= \frac{\mu(r(\eta_1^+))}{\beta^2} \sum_{\sigma_+\xi^\pm, \; \xi^\pm\lambda^\pm} \frac{\sqrt{\mu(r(\sigma_+))}}{\mu(r(\xi^+))}$



$\cdot$



$(\widehat{\xi_1}, \widehat{\xi})$

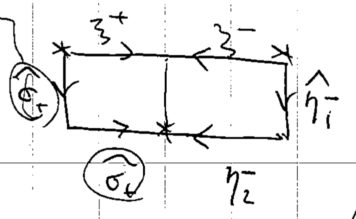
$(\xi\lambda^+, \xi_2\lambda^-)$

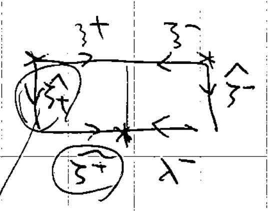
$(\nu_+\widehat{\nu_+}, \nu_-\widehat{\nu_-})$

$\widehat{\xi^+} = \widehat{\lambda^+} \qquad \lambda^+ = \widetilde{\xi^+}$

$f_1 E_{R_1}(f_1(\quad))$

$= \frac{1}{\beta^3} \sum \frac{\sqrt{\mu(r(\sigma_+))}}{\mu(r(\xi^+))} \sqrt{\mu(r(\nu_+))\,\mu(r(\xi_+))}$

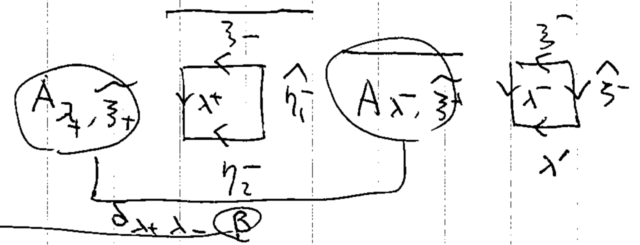




$(\nu_+\widetilde{\nu_+}, \nu_-\widetilde{\nu_-})$

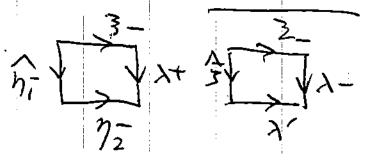
$f_1 E_{\mathcal{B}_1}(f_1(\ \ ))$

$= \frac{1}{\beta^3}\sum \sqrt{\mu(r(\nu_+))}$



$(\nu_+\widetilde{\nu_+}, \xi^-\lambda')$

$= \frac{1}{\beta^2}\sum \sqrt{\mu(r(\nu_+))}$



$(\nu_+\widetilde{\nu_+}, \xi\lambda')$.

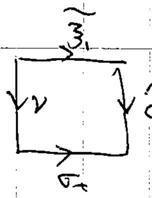
f_1

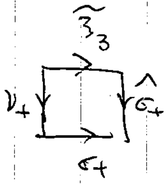
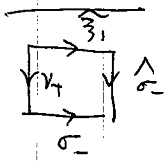
$= \frac{\sqrt{\mu(r(\eta_1))}}{\beta^2}\sum \sqrt{\mu(r(\nu_+))}\ (\nu_+\widetilde{\nu_+}, \eta_1^-\eta_2^-)$

$= \frac{1}{\beta} f_1 \alpha$.

vertical Markov,

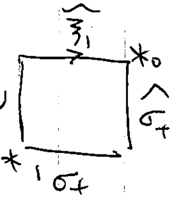
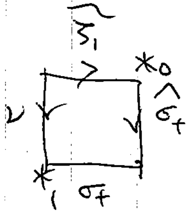
hor. Markov. は明らか.

$\frac{1}{\sqrt{\beta}} \sum_{\nu, \sigma_+} \overline{A_{\nu, \tilde{\xi}_2}}$  $\mu(\sigma(\sigma_+)) = u(\tilde{\xi}_1, \tilde{\xi}_2)$ unitary.

$\sum_{\tilde{\xi}_2} u(\tilde{\xi}_1, \tilde{\xi}_2)^* u(\tilde{\xi}_3, \tilde{\xi}_2) = \sum_{\cancel{\nu_\pm} \sigma_\pm}$   $\mu(r(\sigma_+))\mu(r(\sigma_-))$.

$= \sum_{\nu_+ \cancel{\sigma}} A_{\nu_+, \tilde{\xi}_3} \overline{A_{\nu_+ \tilde{\xi}_1}}$

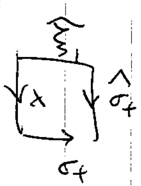
$= \beta \, \delta_{\xi_1, \xi_3}$

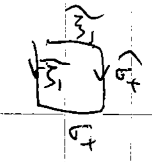
$W'\Bigg($  $\Bigg) = \sum_{\tilde{\zeta}_1} \overline{u(\tilde{\xi}_1, \tilde{\zeta}_1)}$ 

$\sum_{\tilde{\zeta}_1} \frac{A_{\hat{\xi}_1 \tilde{\zeta}_1}}{\beta} A_{\nu \tilde{\zeta}_1} = \delta_{\xi_1, \nu}$

$\lambda = \tilde{\zeta}_1$

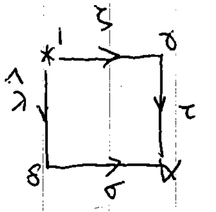
$\frac{A_{\hat{\xi}_1, \tilde{\zeta}_1}}{\sqrt{\beta}}$

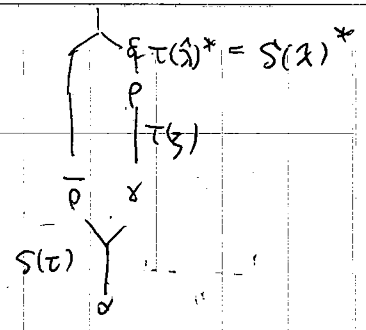
$\sum_{\sigma_+} W'(\quad) \frac{\mu(r(\sigma_+))}{\sqrt{\beta}} = \sum_{\tilde{\zeta}_1} \overline{u(\tilde{\xi}_1, \tilde{\zeta}_1)} \, A_{\underset{\text{fix}}{\nu} \tilde{\zeta}_1} = \frac{1}{\sqrt{\beta}} \sum_{\substack{\cancel{\nu} \sigma_+ \tilde{\zeta}_1 \\ \lambda}} \overline{A_{\underset{\lambda}{\cancel{\nu}} \tilde{\zeta}_1}}$  $\mu(r(\sigma_+)) \; A_{\nu \tilde{\zeta}_1}$

$= \frac{\beta}{\sqrt{\beta}} \sum_{\sigma_+}$  $\mu(r(\sigma_+))$

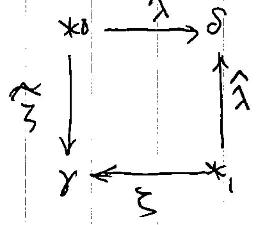
~~Intr~~ Std ⟷ Flatness.

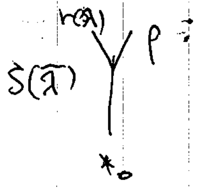
$$\begin{tikzcd}
 & & & & & y \arrow[d, phantom, sloped, "\in"] \\
 & & \mathrm{String}^0\mathcal{G}_0 \arrow[r] \arrow[d, "k_0"] & \mathrm{String}^1\mathcal{G}_0 \arrow[r] \arrow[d, "k_1"] & \cdots \arrow[r] & \mathrm{String}^n\mathcal{G}_0 \arrow[dddd] \\
 & \mathrm{String}^0\mathcal{G}_1 \arrow[r] \arrow[d] & \mathrm{String}^1\mathcal{G}_1 \arrow[r] \arrow[d, "l_0"] & \mathrm{String}^2\mathcal{G}_1 \arrow[r] \arrow[d, "l_1"] & \cdots & \\
\mathrm{String}^0\mathcal{G}_0 \arrow[r] & \mathrm{String}^1\mathcal{G}_0 \arrow[r] & \mathrm{String}^2\mathcal{G}_0 \arrow[r] \arrow[d] & {} & & \\
 & & \vdots \arrow[d] & & & \\
 & & x \in \mathrm{String}^{2m}\mathcal{G}_0 \arrow[rrr] \arrow[d, "k_{2m}"] & & & \mathrm{String}^{2m+n}\mathcal{G}_0 \arrow[d, "k_{2m+n}"] \\
 & & \mathrm{String}^{2m+1}\mathcal{G}_1 \arrow[rrr] & & & \mathrm{String}^{2m+n+1}\mathcal{G}_1
\end{tikzcd}$$

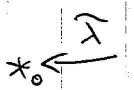




$\xi\, T(\hat{\lambda})^* = S(\lambda)^*$

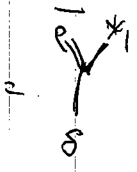


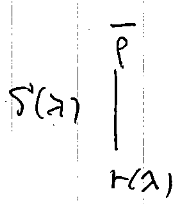




$$S(\lambda) = \sqrt{a(r(\lambda))d(\rho)}\;\; S(\tilde{\lambda})^* \, r(\lambda)(R_\rho)$$

$T(\hat{\lambda}) =$

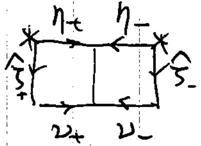




$x = (\xi_+, \xi_-)$, $y = (\eta_+, \eta_-)$

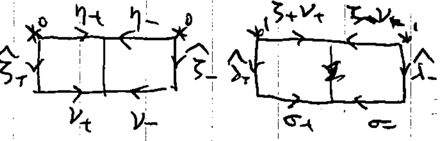
$$\underset{2m,2m+n}{i}(x) = \sum_{\substack{|\lambda| = n \\ \lambda \in \mathcal{G}_0^{\mathrm{edge}}}} (\xi_+ \cancel{\lambda}\lambda, \xi_- \lambda)$$

$(\cancel{\ell}k)^*(y) = \sum$



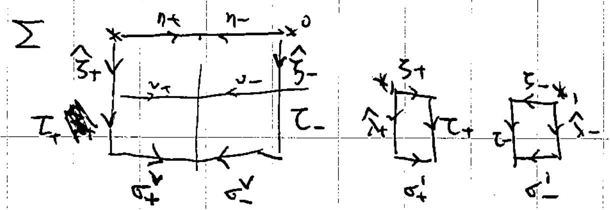
$(\xi_+\nu_+, \xi_-\nu_-)$

$(\ell k)(y) = \sum$



$(\lambda_+\sigma_+, \lambda_-\sigma_-)$

$= \sum$



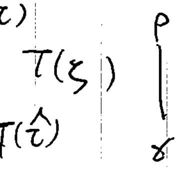
$(\lambda_+\sigma_+, \lambda_-\sigma_-)$

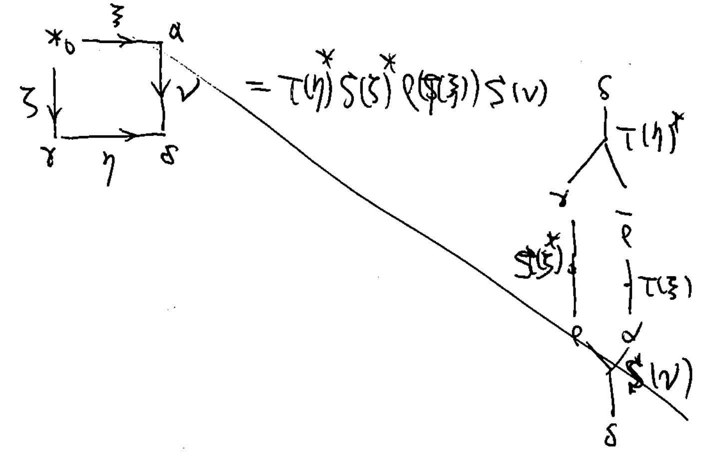
$$S(\sigma)^* T(\hat{\lambda})^* \bar{\rho}(T(\zeta)) S(\tau)$$

$$= S(\sigma)^* \delta(\bar{\rho}(T(\zeta)))$$

$T(\hat{\lambda})^* T(\hat{\tau})$

$T(\zeta)$



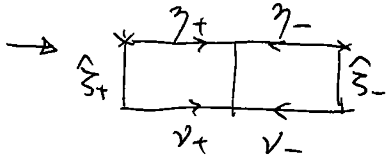


$= \tau(\eta)^* S(\zeta)^* \rho(S(\xi)) S(\nu)$

まず これが $x \in \mathrm{String}^1 \mathcal{G}_1$ について comm. する.

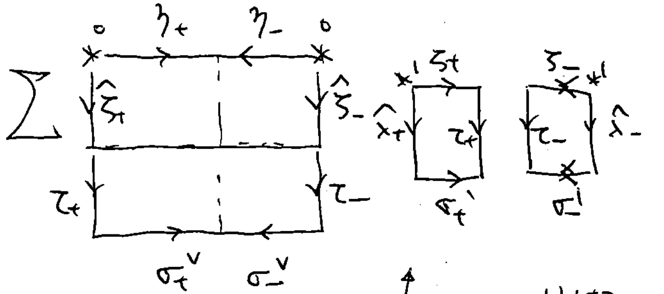
$x = (\xi_+, \xi_-)$

x ~~$\ell k(y) = \sum$~~

→ 

$= \delta_{\xi_+, \xi_-} C_{\eta, \nu}$.

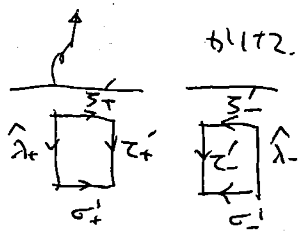
次に $\ell k(y)$ と $\mathrm{string}^2 \mathcal{G}_0$ をみる.

$\sum$ 

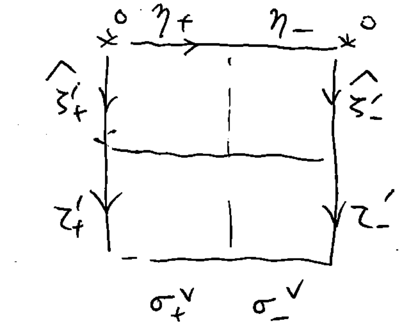
$= \delta_{\lambda_+, \lambda_-} \delta_{\sigma'_+ \sigma'_-} C_{\eta, \sigma^\vee}$.

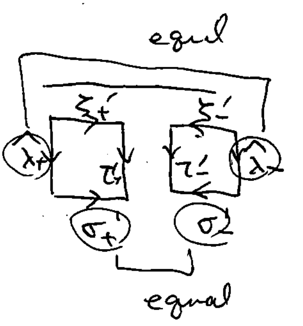
(*)

かけて.



で $(\hat{\lambda}_+, \sigma'_+)$, $(\hat{\lambda}_-, \sigma'_-)$ で sum.

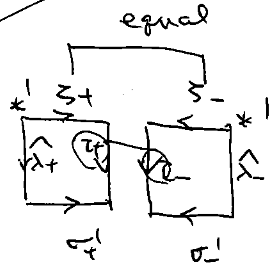
⇝ 

$= \sum$ 

$C_{\eta, \sigma^\vee}$

$= \delta_{\xi'_+, \xi'_-} \delta_{\tau'_+ \tau'_-} C_{\eta, \sigma^\vee}$.

逆に. これをみとめると

LHS(*) $= \sum C_{\eta, \sigma^\vee}$ 

$= \delta_{\lambda_+ \lambda_-} \delta_{\sigma'_+ \sigma'_-} C_{\eta, \sigma^\vee}$.

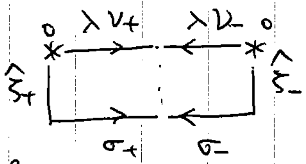
これから. これがいえる.

$k(\eta) = \sum C^{k}_{\eta,\nu} (\overset{\overset{1}{\downarrow}}{\xi}\nu_+ , \xi\nu_-)$

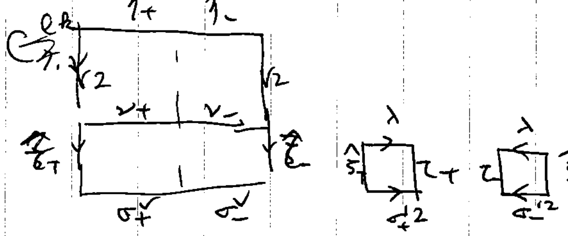
$lk(\eta) = \sum C^{lk}_{\eta,\nu} (\underset{\underset{2}{\uparrow}}{\lambda}\nu_+ , \lambda\nu_-) \in \mathrm{string}^{n+2} g_0$

$klk(\eta) \in \mathrm{strng}^{n+3} g_1$

$= \sum C^{lk}_{\eta,\nu}$



$(\zeta_+\sigma_+ , \zeta_-\sigma_-)$

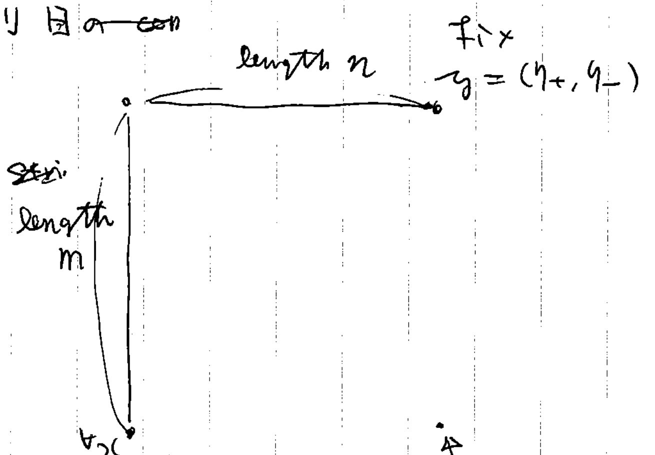


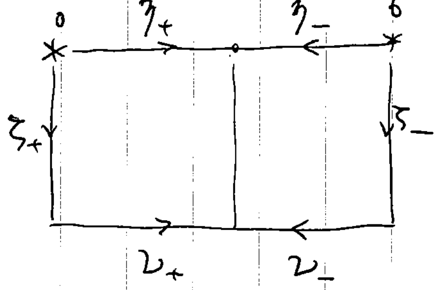
$(\zeta_+\sigma_+ , \zeta_-\sigma_-)$.

同様の考察によって

↓ length 3 の(ξ_+ , ξ_-) のケースがでる

0列目の ~~con~~



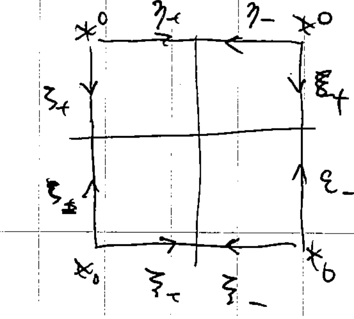


$= \delta_{\zeta_+ , \zeta_-} \; \exists \; C^{\mathrm{length}(\zeta)}_{\eta,\nu}$

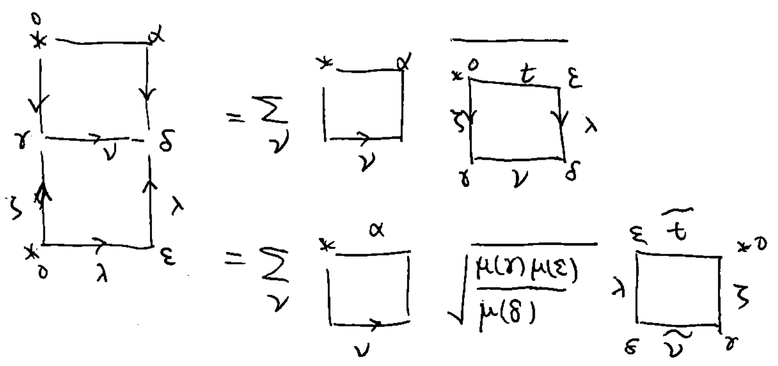
⟺

ψ ==> commute.

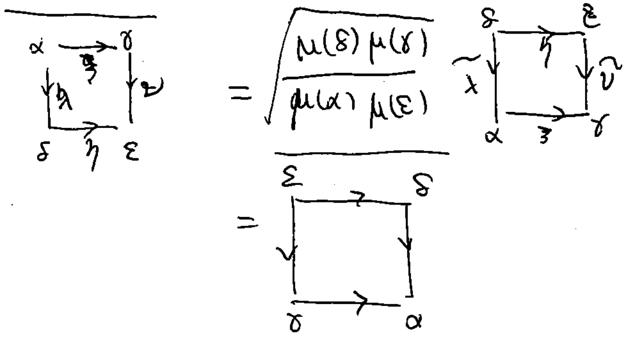
⟹



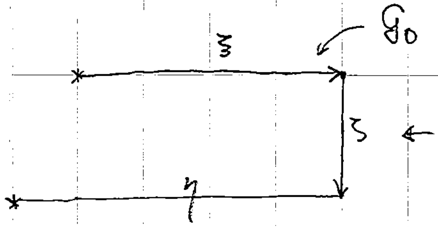
$= \delta_{\zeta_+ , \varepsilon_+} \delta_{\zeta_- \varepsilon_-} \; C_{\eta,\xi}$.



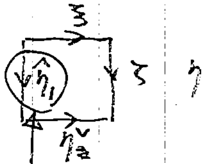
$= \sum_{\nu}$
$= \sum_{\nu}$
$\sqrt{\frac{\mu(\gamma)\mu(\varepsilon)}{\mu(\delta)}}$
α γ δ ε ζ λ ν t $\tilde{t}$ $\tilde{\nu}$



$= \sqrt{\frac{\mu(\delta)\mu(\gamma)}{\mu(\alpha)\mu(\varepsilon)}}$
=
α γ δ ε η ξ $\tilde{\lambda}$ $\tilde{\nu}$



← good (horizontal.
length 1

$\xi\cdot\zeta := \sum_{\eta}$


initiation.

$$\begin{tikzcd} r(\zeta):\ \mathrm{Path}^n_{*_0} g_0 \arrow[r] \arrow[d, hook] & \mathcal{H}^{n+1}_{*_1} g_1 \\ \mathcal{H}^n_{*_0} g_0 \arrow[ur] & \end{tikzcd}$$

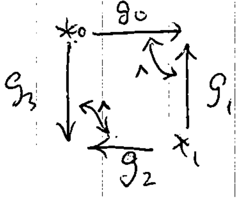
isometry.

$$\sum_{\substack{\zeta \\ |l(\zeta)|=1}} r(\zeta) r(\zeta)^* = 1.$$

$$\to \begin{tikzcd} \mathrm{String}^n_{*_0} g_0 \arrow[d, "k_n"] \\ \mathrm{String}^{n+1}_{*_1} g_1 \end{tikzcd}$$

$$k_n(\cdot) = \sum_{\zeta} r(\zeta)\cdot r(\zeta)^*$$

$$\mathrm{Path}^1_{*_0} g_0 \overset{\wedge}{\longleftrightarrow} \mathrm{Path}^1_{*_1} g_1$$

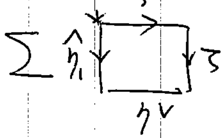
~~&~~


$r(\zeta)\,\xi =: \xi\cdot\zeta$ とかく.

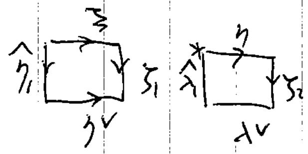
$$\mathrm{Path}^{n+1}_{*_1} g_1 \xrightarrow{r(\zeta)} \mathrm{Path}^{n+1}_{*_0} g_0$$

$$\begin{tikzcd} \mathrm{String}^{n+1}_{*_1} g_1 \arrow[d, "l_n"] \\ \mathrm{String}^{n+2}_{*_0} g_0 \end{tikzcd}$$

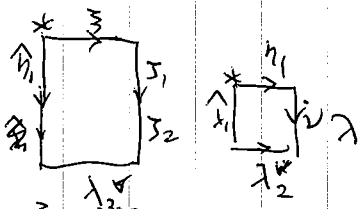
とすると.

$(\xi\cdot\zeta_1)\cdot\zeta_2 = \sum$


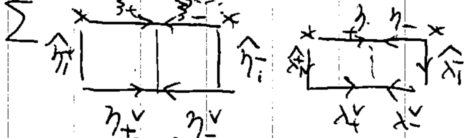
$\eta\cdot\zeta_2$

$= \sum$


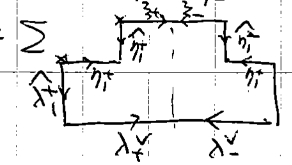
λ

$\Big(= \sum$


λ $\Big)$

$lk(\xi_+, \xi_-) = \sum$


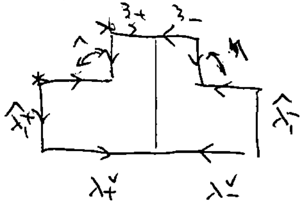
$\hat{\lambda}_i(\lambda_+, \lambda_-)$

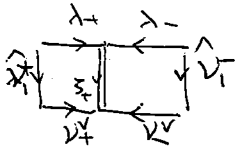
$= \sum$


$\hat{\lambda}_i\ (\lambda_+, \lambda_-)$

$k\ell k(\xi_+,\xi_-)$

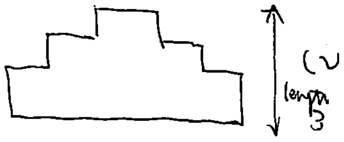
$= \sum$





(ν_+,ν_-)

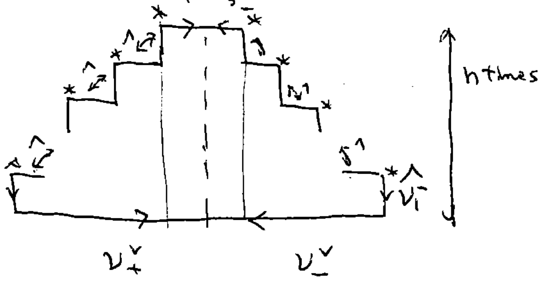
$= \sum$



(ν_+,ν_-)

$\overbrace{k\ell\cdots}^{n \text{ times}}(\xi_+,\xi_-)$

$= \sum$



(ν_+,ν_-)

~~Pat~~

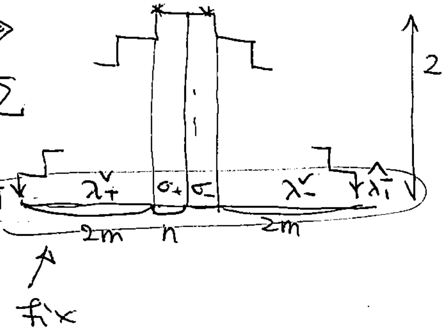
$$\begin{tikzcd}
 \arrow[r, dash] & \mathrm{String}^n_{*_0} g_0 \arrow[d, "(\ell k)^m"] \\
\mathrm{String}^{2m}_{*_0} g_0 \arrow[u, dash] \arrow[r, dash] & \mathrm{String}^{2m+n}_{*_0} g_0
\end{tikzcd}$$

$\cup$

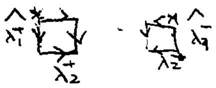
x

$\forall x \quad x\,(kk)^m(\eta) = (\ell k)^m(\eta)\,x.$

$\Longleftrightarrow \sum$

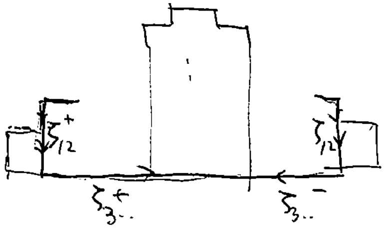


$= \delta_{\lambda_+,\lambda_-}\, C_{\xi.\sigma}$

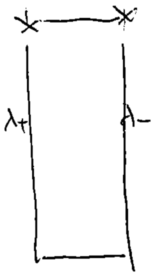


かけて和R

$\Rightarrow$
$\Leftarrow$



$= \delta_{\zeta^+\zeta^-}\; C_{\xi,\sigma} \Longleftrightarrow \cdots$



$= \delta_{\lambda_+,\lambda_-}$

$C_{\xi,\sigma}.$

$(\ell k)^m(\xi_+,\xi_-) = \sum (\xi_+\cdot\zeta\ ,\ \xi_-\cdot\zeta)$

$\left(= \sum \pi(\zeta)(\xi_+,\xi_-)\pi(\zeta)^*\cdot\right)$

$x = (\eta_+,\eta_-) \in \mathrm{Strng}^{2m}_{*_0}(\mathcal{G}_0)$

$x\ (\ell k)^m(\xi_+,\xi_-) = (\qquad)\ x$

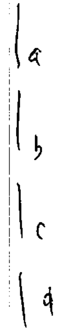
$\forall\xi$

$\Leftrightarrow \pi(\zeta_+)^* x\,\pi(\zeta_-)\ (\xi_+,\xi_-) = \pi(\zeta_+)^* x\,\pi(\zeta_-)^{(\xi_+,\xi_-)}$

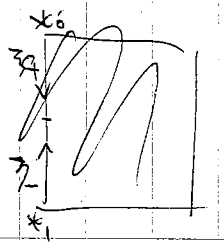
$\Leftrightarrow \pi(\zeta_+)^* x\,\pi(\zeta_-) \in (\mathrm{String}^n_{*_0}\mathcal{G}_0)' \subset B(H^n_{*_0}\mathcal{G}_0)$

$\parallel$

$Z(\mathrm{Strng}^n_{*_0}\mathcal{G}_0)$

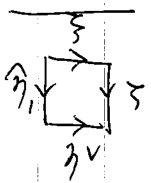


$\forall$ ~~ξ_+~~, ξ_-



$\pi(\zeta_+)^* = \pi(\zeta_1^+\cdots\zeta_{2m}^+)^* = \pi(\zeta_1^+)^*\cdots\pi(\zeta_{2m}^+)^*$

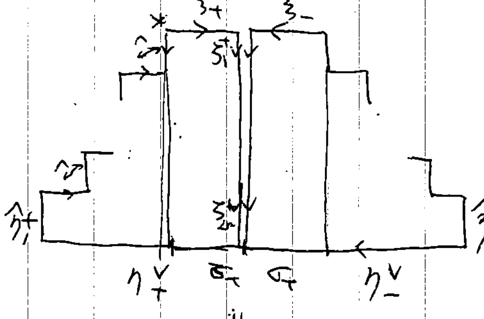
$\pi(\zeta)^*\eta = \sum$



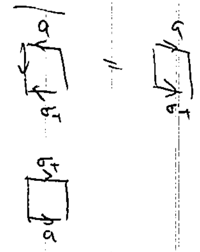
ξ

$\pi(\zeta_1^+)^*\cdots\underline{\pi(\zeta_{2m}^+)^*(\eta_+,\eta_-)}\cdots\cdots\pi(\zeta_1^-)$

$=\sum$



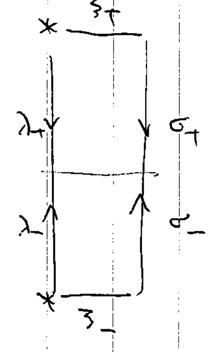
(ξ_+,ξ_-)



$\parallel$

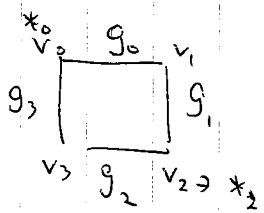
$\delta_{\xi_+,\xi_-}\ C_\zeta(\pi(\xi_+)))$

$\Leftarrow$

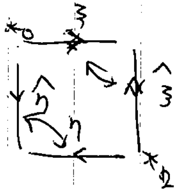


$= \delta_{\xi_+,\xi_-}\ C_{\lambda,\sigma}.$

Summary

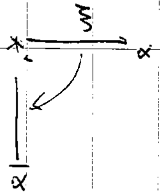


- Connected graphs
- $\mu(x)$ β.
- initiation map



W() flat biunitary connection

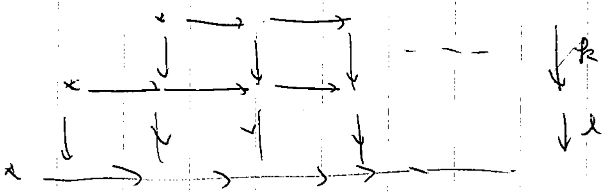
Para grp



in G_0

$\exists$ contragradient map

→ Std. lattice.



rigid.
→ C^* − 2 category $\mathcal{C}^\rho$

テンソル
これの分解則

§

$$\begin{tikzcd}
A_{0n} \arrow[rr, "\theta_{0n}"] \arrow[dr, "i_n"] \arrow[dd, "k_n"] & & P_{0n} \arrow[dr] \arrow[dd] & \\
& A_{0n+1} \arrow[rr, "\theta_{0n+1}"] \arrow[dd, "k_{n+1}"] & & P_{0n+1} \arrow[dd] \\
A_{1n} \arrow[rr, "\theta_{1n}"] \arrow[dr, "j_n"] & & P_{1n} \arrow[dr] & \\
& A_{1n+1} \arrow[rr, "\theta_{1n+1}"] & & P_{1n+1}
\end{tikzcd}$$

P_{0n}, P_{1n}. Path alg.s

generated by

(ξ_+, ξ_-)

- $r(\xi_+) = r(\xi_-)$
- $|\xi_+| = |\xi_-| = n$.
- ξ_+, ξ_- は $-A_{0n} \to A_{0n+1} \to$
の Bratteli 図 の edge

θ_{0n}. θ_{1n} の構成 をふり返る.

$\mathrm{Irr}\, A_{0n} = \{\pi_x\}_{x \in I_n}$ を fix.

$\pi_x : A_{0n} \to B(H_x)$.
irred.

$\ker \pi_x = A_{0n} z_x^{\perp}$

$\mathrm{Irr}\, A_{1n} = \{\sigma_x\}_{x \in J_n}$

$\rightsquigarrow \quad A_{0n} \xrightarrow{\Phi_n} \bigoplus_{x \in I_n} B(H_x)$. $\qquad A_{1n} \xrightarrow{\Psi_n} \bigoplus_{y \in J_n} B(H_y)$

次に.

$A_{0n} \xrightarrow{i_n} A_{0n+1}$. $\qquad A_{1n} \xrightarrow{j_n} A_{1n+1}$.

から作られる

$\pi_y \cdot i_n \quad \pi_y \in \mathrm{Irr}\, A_{0n+1}$

$\mathrm{Mor}(\pi_x, \pi_y \cdot i_n)$ は Hilb. sp.

$$\begin{array}{ccc} \bigoplus_{x \in I_n} \mathrm{Mor}(\pi_x, \pi_y \cdot i_n) \otimes H_x & \longrightarrow & H_y \\ \rotatebox[origin=c]{-90}{$\in$} & & \rotatebox[origin=c]{-90}{$\in$} \\ T \otimes v & \longmapsto & Tv \end{array}$$

内積を保つ unitary map.

i_n のB図の各 edge

$$\{X \xrightarrow{e} Y\}_e \quad s(e)=X,\ r(e)=Y.$$

に対して $\{T(e)\}_e$ ONB of $\mathrm{Mor}(\pi_X, \pi_Y\cdot i_n)$

を選んでおく. ↑ 自分で pick up しておく

Long ~~ed.~~ path

$$e = e_1 e_2 \cdots e_n$$

に対して

$$T(e) = T(e_n) \cdots T(e_1) \in \mathrm{Mor}(\pi_{s(e_1)}, \pi_{r(e_n)} \cdot i_n \cdots i_0)$$

とおくと. $\{T(e)\}_{|e|=n+1,\ r(e_n)=Y}$ は H_Y の ONB を与えている.

$$\underset{x\in I_n}{B(H_x)} = \mathrm{span}\{T(e_+)T(e_-)^* \mid r(e_+)=r(e_-)=x\}$$

$$\cong P^x_{0n} = \mathrm{span}\{(e_+, e_-) \mid r(e_+)=r(e_-)=x\}$$

$$\begin{tikzcd}
A_{0n} \arrow[r, "\Phi_n"] \arrow[d, "i_n"] & \bigoplus_{x\in I_n} B(H_x) \arrow[r, "\sim"] & \bigoplus_{x\in I_n} P^x_{0n} \arrow[d, "I_n"] \\
A_{0n+1} \arrow[rr, "\varphi_{n+1}"] & & \bigoplus_{Y\in I_{n+1}} P^Y_{0n+1}
\end{tikzcd}$$

$$r(e_+)=r(e_-)=X$$

$$I_n((e_+, e_-)) = \Phi_{n+1}\left(i_n \overset{\Phi_n^{-1}}{\pi_X^{-1}}(T(e_+)T(e_-)^*)\right)$$

$$= \bigoplus \pi_Y(i_n \pi_X^{-1}(T(e_+)T(e_-)^*))$$

$$= \bigoplus_{Y}\sum_{Z\in I_n} T\,\pi_Z(\pi_X^{-1}(T(e_+)T(e_-)^*))T^*$$

$$T \in \mathrm{ONB}(\pi_Z, \pi_Y\cdot i_n)$$

$$= \bigoplus_{Y}\sum_{T} T\,T(e_+)T(e_-)^*T^*.$$

$$= \bigoplus_{Y}\sum_{e} \underbrace{T(e)T(e_+)}_{T(e_+e)}\underbrace{T(e_-)^*T(e)^*}_{T(e_-e)}$$

$$e: X \to Y$$

$$= \sum_{s(e)=X} (e_+e, e_-e)$$

$$\begin{tikzcd}
 & H_x & H_y & \\
\arrow[r] & A_{0n} \arrow[r] \arrow[d, "k_n"] & A_{0n+1} \arrow[r] \arrow[d, "k_{n+1}"] & \cdots \\
\arrow[r] & A_{1n} \arrow[r] & A_{1n+1} \arrow[r] & \cdots \\
 & H_z & H_w &
\end{tikzcd}$$

base を決めている．

$A_{0n} \xrightarrow{k_n} A_{1n}$ の B図 の edge $\overset{\lambda}{\cancel{\xi}} : x \to z$ に対し

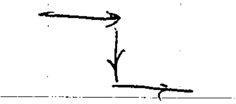
$S(\lambda) \in \mathrm{ONB}(\pi_x \; \pi_z k_n)$

を対応させる．

$\leadsto \{S(\lambda)T(\overset{k_{0n}\text{ edges}}{e})\}$ $s(\lambda) = r(e) = x$, $\forall x$
$r(\lambda) = z$.

が H_z の base を与える．

$\leadsto \{T(\overset{\text{下段 } k_{01}}{\nu})S(\lambda)\ T(e)\}$ ONB of $H_{r(\nu)}$



同様に

$\{\cancel{S}(\eta)\ T(e')\}$

$S(\eta) \in \mathrm{ONB}(\pi_{\cancel{s}(\eta)}, \pi_{r(\eta)} \cdot k_{n+1})$

も $H_{r(\eta)}$ の ONB を与える．

そこで

$$\begin{tikzcd}
\cdot \arrow[r, "\xi"] \arrow[d, "\lambda"'] & \cdot \arrow[d, "\eta"] \\
\cdot \arrow[r, "\nu"'] & \cdot
\end{tikzcd} := T(\lambda)^* T(\nu)^* S(\eta) T(\xi) \in \mathbb{C}1_\xi$$

と定める．

これは $H_{r(\nu)}$ の

$\{S(\eta)T(e)\ T(e_0)\}$

と

$\{T(\nu)\ S(\lambda)\ T(e_0)\}$

の 交換行列 を 与える．

k_n を connection の元で記述する.

$$k_{n+1}(\quad \pi_y^{-1}((e_+\xi_+, e_-\xi_-)))$$

$$= k_{n+1}(\pi_y^{-1}(T(e_+\xi_+, e_-\xi_-)))$$

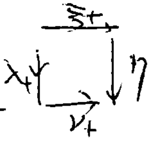
$$= k_{n+1}(\pi_y^{-1}(T(\xi_+)T(e_+)T(e_-)^*T(\xi_-)^*))$$

$$\begin{tikzcd} A_{0n+1} \arrow[d] & \bigoplus P_y^{n+1} \arrow[l] \\ A_{1n+1} \arrow[r, dash] & \bigoplus P_w^{n+1} \end{tikzcd}$$

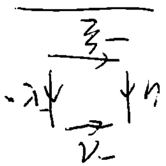
$$\to \bigoplus_w \pi_w k_{n+1} \pi_y^{-1}(\quad -//- \quad)$$

$$= \bigoplus_w S(\eta)\pi_y \cancel{\pi_y} T(\xi_+)T(e_+)T(e_-)^*T(\xi_-)^*)\, S(\eta)^*$$

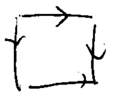
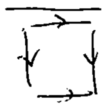
$\eta : y \to w$

$= \sum_w$ 

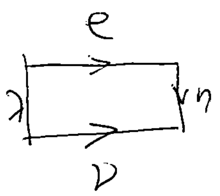
$\eta\; T(\nu_+)S(\lambda_+)T(e_+)\cdot$ 

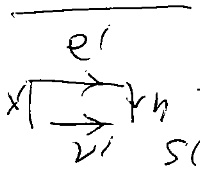


$\eta\; T(e_-)^*S(\lambda)^*T(\nu_-)^*$

$= \sum_W$   $(e_+\lambda_+\nu_+, e_-\lambda_-\nu_-)$

=

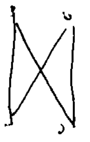
$= \sum$ 

$\eta\; T(\nu)S(\lambda)$ 

$\eta\; T(\nu')^*S(\lambda')^*$

$\|$

$T(\nu)$ と書いてよい

(もし $A_{00} \xrightarrow{h_0} A_{10}$ が  のように single edges から なりたっていたら、$\mathcal{S}(\nu)$ から 決まるから)

//

Basic Extensions.

$$\begin{tikzcd} A \arrow[r, shift left, "i_A"] & B \arrow[l, shift left, "\phi_A"] \arrow[r, "\cancel{i_{\mathcal{C}}}"] & \mathcal{C} \end{tikzcd}$$ ($\cancel{\mathcal{C}}^*$-algs)
W^*

$\varphi \in A_{*+}^*$ state. faithful.

$\psi := i_A(\varphi) = \varphi \circ \phi_A$.

$(\pi_\psi, \cancel{H_\varphi} L^2(B, \psi), \xi_\psi)$.
GNS

$(\pi_\varphi, L^2(A, \varphi), \xi_\varphi)$

$$\begin{array}{ccc} v_A : L^2(A, \varphi) & \longrightarrow & L^2(B, \psi) \\ \cup & & \cup \\ \pi_\varphi(x)\xi_\varphi & \longmapsto & \pi_\psi(x)\xi_\psi \end{array}$$

isometry

$$e_A := v_A v_A^* \in \pi_\psi(i_A(A))' \cap B(L^2(B, \psi))$$

$$e_A \pi_\psi(x) e_A = v_A \underset{\underset{\pi_\varphi(\phi_A(x))}{\parallel}}{v_A^* \pi_\psi(x) v_A} v_A^* = \pi_\psi(\phi_A(x)) e_A.$$

J_ψ conj.

$\langle B, A \rangle := \pi_\psi(B) \vee \{e_A\}''$.

$= J_\psi \pi_\psi(i_A(A))' J_\psi$ ← これをみると、
ψのとりかえで
影響
しない
ことがわかる

$$\begin{tikzcd} A \arrow[r, shift left, "i_A"] & B \arrow[l, shift left, "\phi_A"] \arrow[r, "\pi_\psi"] & \langle B, A \rangle \end{tikzcd}$$
↑ Basic ext という.

i_A の ~~Path~~String alg ~~[illegible]~~ 考えだから.

$\langle B, A \rangle$ の それを 得ることを 考える.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d, equal] & A \arrow[r, "\iota_A"] \arrow[d, "\theta_A", "\wr"'] \arrow[rd, phantom, "\circlearrowright"] & B \arrow[l, bend right, "\Phi_{\iota_A}"'] \arrow[r, "\iota_B"] \arrow[d, "\theta_B", "\wr"'] & C \arrow[l, bend right, "\Phi_{\iota_C \#}"'] \\
\mathbb{C} \arrow[r] & \mathrm{Strg}_A \arrow[r, "\bar{\jmath}_A"] & \mathrm{Strg}B \arrow[l, bend left, "\Phi_{\jmath_A *}"] \arrow[r] & C'
\end{tikzcd}$$

$\jmath_A$ は具体的

} 図[illegible]んで置い
まとめる必要あり.

$\mathrm{Irr}\,A = \{x\}_{x \in J_A}$ $\quad$ $\mathrm{Irr}\,B = \{Y\}_{Y \in I_B}$.

$C \curvearrowright L^2(\mathrm{Strg}\,B) = \mathrm{Span}\{ (\xi, \eta) \ | r(\xi) = Y\}_{Y}$.

これ迄の submodule として irr a list をつくる.

~~(零x~~

$$\mathrm{End}_C\, L^2(\mathrm{Strg}\,B) = C'$$
$$= (JA'J)' = JAJ.$$

各 irr は $x \in \mathrm{Irr}A$ で label される.

$p_x = J(\underset{\substack{x \Uparrow \\ \mathrm{Strg}\,A}}{\xi}, \xi)J \quad r(\xi_x) = x$.

が irr. comp となる.

$$H_x^C := p_x L^2(\mathrm{Strg}\,B)$$
$$= L^2(\mathrm{Strg}) \otimes (\xi_x \xi_x)$$
$$= \mathrm{span}\{(\zeta, \xi_x \nu) \mid {}^\forall \zeta, {}^\forall \underset{\substack{\Uparrow \\ s(\nu) = x \\ r(\zeta) = r(\nu)}}{\nu}\}\ \}$$

$(\pi_x^C, H_x^C) \longleftarrow\!\!| \; x \in \mathrm{Irr}\,C$.

$\mathrm{Strg}_B \xrightarrow{\iota_B} C \qquad$ a Bratteli 図

$(\pi_Y^B, \; \pi_x^C \cdot \iota_B)$ $\quad$ をしらべる

$\pi_x^C \iota_B((\underset{\substack{\Uparrow \\ \mathrm{Strg}\,B}}{\eta}, \xi)) \quad (\zeta', \xi_x \nu)$

$= \delta_{\xi, \zeta'} (\eta, \xi_x \nu)$

$$C L^2(\mathrm{Strg}_B) = \sum^{\oplus} H_x^C \cdot \#\{\xi \mid r(\xi) = x\}$$
$$= \sum^{\oplus} H_x^C \cdot \dim H_x^A.$$

π_Y^B : Strig $B \longrightarrow B(H_Y^B)$

$H_Y^B = \mathrm{Path}_{B,Y}$

$= \mathrm{span}\{\zeta \mid r(\zeta) = Y\}$.

$(\xi, \eta)\zeta = \delta_{r(\eta), Y}\ \xi$.

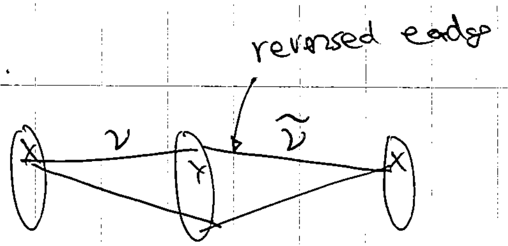
各 edge $\nu : X \longrightarrow Y$ に対して isometry.

$$\begin{tikzcd} S_\nu : H_Y^B \arrow[r] & H_X^C \\ \zeta \arrow[u, phantom, "\in"] \arrow[r, mapsto] & (\zeta, \xi_X \nu) \cdot \dfrac{1}{\tau(r(\nu))^{\frac{1}{2}}} \arrow[u, phantom, "\in"] \end{tikzcd}$$

$$\left(\begin{aligned} &\langle (\zeta, \xi_X \nu), (\zeta', \xi_X \nu) \rangle \\ &= \delta_{\zeta, \zeta'}\ \varphi((\xi_X \nu, \xi_X \nu)) \\ &= \delta_{\zeta, \zeta'}\ \tau(r(\nu)). \end{aligned} \right)$$

$\rightarrow (\pi_Y^B, \pi_X^C \cdot i_B) = \mathrm{span}\{S_\nu\}_{\nu : X \rightarrow Y}$.

ここで.



という graph を考えて. $\tilde{\nu} : Y \rightarrow X$

$T(\tilde{\nu}) \in (\pi_Y^B, \pi_X^C \cdot i_B)$

を $T(\tilde{\nu}) := S_\nu$ で定める.

すると. $H_X^C = \mathrm{span}\{T(\tilde{\nu}) T(\zeta)\}_{s(\nu) = X}$

とできる. こうして. Path_C を つくることができる.

次に C. と. ~~Path~~ Str$_C$ を 比較する.

$\downarrow$ $\downarrow$

$L^2(\mathrm{strig}_B)$. Path_C.

$$\begin{tikzcd}
\mathrm{Path}_C \arrow[rr, dash] & & L^2(\mathrm{String}. \arrow[dl, bend left, "\text{unitary}"'] \\
\mathrm{Path}_{C,x} & H_x^C \arrow[l, "U_x"'] & \\
\tau(r(\nu))^{\frac{1}{2}}\xi\tilde{\nu} \arrow[u, phantom, "\in"] & (\xi, \xi_x\nu) \arrow[l, mapsto] \arrow[u, phantom, "\in"] &
\end{tikzcd}$$

$\nu : X \to \cancel{r}(\xi).$

$\xi \in \mathrm{Path}_B.$

$$\begin{tikzcd}
C \arrow[r] & \bigoplus_X B(H_x^C) \arrow[r] & \cancel{\mathrm{String}}\, C. \\
\mathrm{Strg}_B = \bigoplus_Y B(H_Y^B) \arrow[u, "i_B"] \arrow[ur] & &
\end{tikzcd}$$

$\pi_X^C(i_B(\xi,\eta).)\ (\zeta\cancel{\ }, \xi_x\nu) = \delta_{\eta,\zeta}\ (\xi\cancel{\ }, \xi_x\nu).$

$\quad = T(\tilde{\nu})T(\zeta)$

$i_B^{s\dagger}(\xi,\eta)\ T(\tilde{\nu})T(\zeta) \neq i_B^{s\dagger}(\xi,\eta)\ T(\zeta\tilde{\nu})$

$= \delta_{\eta,\zeta}\ T(\xi\tilde{\nu})$

$= \delta_{\eta,\zeta}\ T(\tilde{\nu})\,T(\xi)1$

$\rightsquigarrow \pi_X^C(i_B(\xi,\eta)) = i_B^{\xi\dagger}(\xi.\eta).$

$\pi_C(e_A) = e_A \cdot (\overset{\xi\rho}{\xi}. \xi_x\nu)\ 1$

$= \delta_{\rho,\nu}\sqrt{\lambda}\ \dfrac{\mu(r(\nu))}{\mu(X)}\ (\xi.\xi_x)$

$\pi_C(e_A)(\cancel{\xi\rho}\xi_x\nu) = \sqrt{\lambda}\ \delta_{\rho,\nu}\ \dfrac{\mu(r(\nu))}{\mu(X)}^{\frac{1}{2}} \sum_\sigma (\xi\sigma, \xi_x\sigma)$

$\quad = \tau(\nu)^{\frac{1}{2}}\, T(\hat{\nu})T(\xi\rho) = T(\xi\rho\hat{\nu})$

$\quad\ \sum_\sigma T(\tilde{\sigma})\ T(\xi\sigma)1\,\tau(\sigma)^{\frac{1}{2}} = \sum_\sigma T(\xi\sigma\tilde{\sigma})$

$\sqrt{\lambda}\sum \dfrac{\sqrt{\mu(r(\sigma))\mu(r(w))}}{\mu(\cancel{r}(\xi))}.(\xi\sigma\tilde{\sigma}, \xi w\hat{w}).$

$\curvearrowright\ \xi\rho\tilde{\nu}$

$\rho = w = \nu : X \to Y$

$= \sqrt{\lambda}\ \sum_\sigma \dfrac{\sqrt{\mu(\sigma))}\sqrt{\mu(r(\nu)}}{\mu(\cancel{r(\xi)}\,X)}\ \xi\sigma\tilde{\sigma}$

よって $\pi_C(e_A) = \sqrt{\lambda}\sum_{\xi\sigma w} \dfrac{\sqrt{\mu(r(\sigma))\mu(r(w))}}{\mu(r(\xi))}(\xi\sigma\tilde{\sigma}, \xi w\tilde{w})$

1

Lem. basic ext

$$\begin{tikzcd}
A \arrow[r, "i_A", "\phi_A"'] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\theta_B"] & C \arrow[d, "\exists C"] & e_A \arrow[d, mapsto] \\
\mathrm{St}_A \arrow[r, "j_A", "\phi_{j_A}"'] & \mathrm{St}_B \arrow[r, "j_B", "\phi_{j_B}"'] & \mathrm{St}_C & e_A^{st}.
\end{tikzcd}$$

Graph は 折り返しで与えられる.

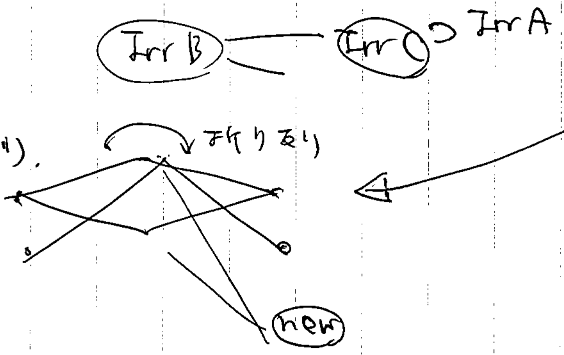
∟

Lem.

$$\begin{tikzcd}
A \arrow[r, "i_A", "\phi_A"'] \arrow[d, "\theta_A"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "i_C", "\phi_B"'] \arrow[d, "\theta_B"] & C \ni e_A \\
\mathrm{St}_A \arrow[r, "\phi_B"'] & \mathrm{St}_B &
\end{tikzcd}$$

e_A Markov. とする.

とすると. ある graph



となり.

となっていて.

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\theta_B"'] \arrow[dr, phantom, "\circlearrowleft"] & C \ni e_A \arrow[d, "\theta_C"] \\
\mathrm{St}_B \arrow[r, leftrightarrow] & \mathrm{St}_C \ni e_C^{st}.
\end{tikzcd}$$

となる

∟

pf.

z_A

$$C = B e_A B + C z_A^{\perp}$$ に分けると.

$$\mathrm{Irr}\, C = \mathrm{Irr}\, B e_A B \sqcup \mathrm{Irr}\, C(z_A^{\perp})$$ と [illegible].

$\mathrm{Irr}\, B e_A B \ni X$, (π_X^C, H_X^C) を z_A で cut してかぶせる.

$\mathrm{Irr}\, C(z_A^{\perp}) \ni$ ~~[illegible]~~ $\{\pi_W, H_W^C$

Graph.

(π_Y, π_X^C, i_C) (π_Y, π_W^C, i_C)

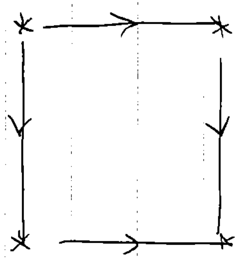
→ $\pi_X^C(i_B(\xi, 0)) = i_B^{st}(\xi, \eta)$

∎

λ-lattice から connection をつくることができる.

そして. Standard を与えると. 2 圏が

来るが flat でなくてはならない.



$= \delta_{-,-} \quad \delta_{-.-}$

逆に flat と仮定する.

$$\begin{tikzcd}
 & \mathrm{Path}_{00} \arrow[r] \arrow[d] & \mathrm{Path}_{01} \arrow[r, "\overset{\xi}{\cap}"] \arrow[d] & \mathrm{Path}_{02} \\
\mathrm{Path}_{10} \arrow[r] & \mathrm{Path}_{11} \arrow[r] & \mathrm{Path}_{12} \arrow[r] & \cdots
\end{tikzcd}$$

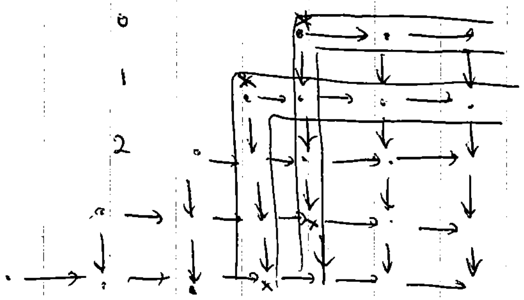
$\sum_{\lambda} \xi\lambda \sqrt{\frac{d(r(\lambda))}{d(\rho)d(r(\xi))}}$

$$\begin{tikzcd}
 & {}^{St}\mathrm{Path}_{0,A} \arrow[r] \arrow[d] & {}^{St}\mathrm{Path}_{01} \arrow[r] \arrow[d, "k_1"] & \cdots \\
{}^{St}\mathrm{Path}_{0,B} \arrow[r] & {}^{St}\mathrm{Path}_{1,1} \arrow[r] & {}^{St}\mathrm{Path}_{1,2} \arrow[r] & \cdots
\end{tikzcd}$$

Path_{01}^{h}

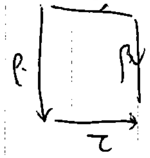
$$k_1(\xi, \xi) = \sum_{\lambda} (\xi\lambda, \xi\lambda)$$

$$= \sum \nu \overset{\xi}{\boxed{}}_{\eta} \lambda \; (\nu\eta, \nu\eta) \; \overline{\boxed{}}$$



この図から flat と std. が分かる.

$$(\rho_+, \rho_-) = \sum_{\tau} (\rho_+\tau, \rho_-\tau)$$

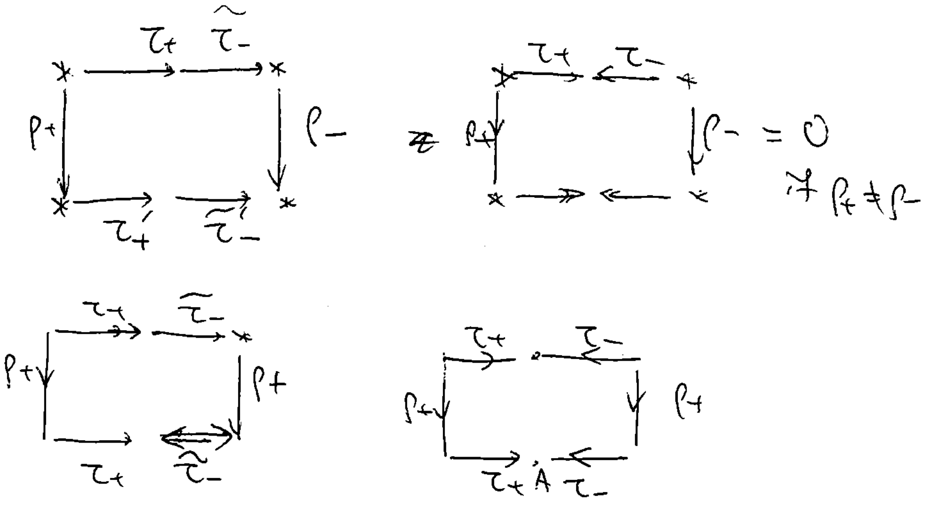


$$\| $$

$$\sum_{\tau\alpha,\gamma} \rho_+ \overset{\alpha}{\boxed{}}_{\tau} \beta \; (\alpha\beta, \gamma\delta) \; \overline{\rho_- \overset{\gamma}{\boxed{}}_{\tau} \delta}$$

$$(1) \;\|$$

$$\sum_{\tau} c_{(\rho_+,\rho_-)(\gamma.\delta)} \; (\tau\beta, \tau\delta)$$



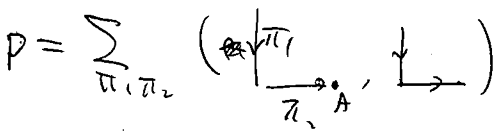
こうして。

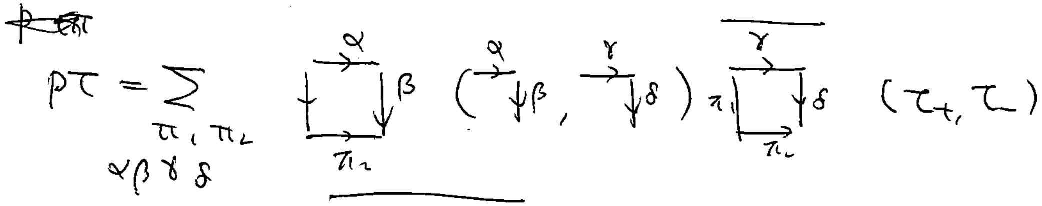
connection flat

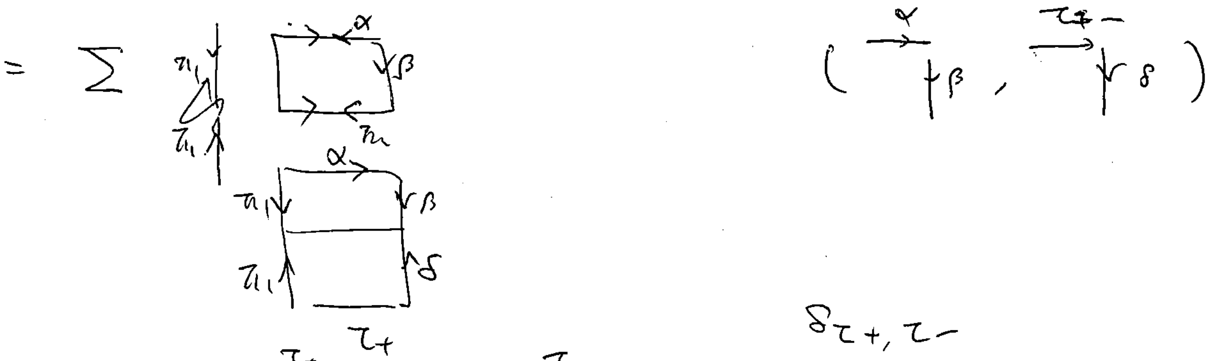
⟷

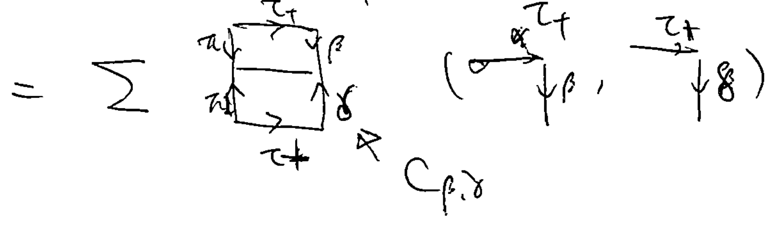
λ-lattice std

の対応がつく。









Connection

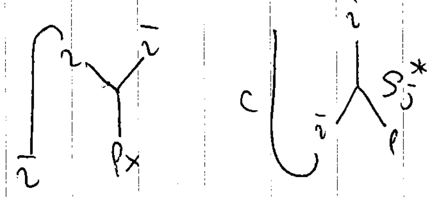
$(\bar{\iota}, t\otimes\bar{\iota}) = (\bar{\iota}, \alpha_t\bar{\iota}) = \mathbb{C}$.

$$\begin{tikzcd}
t \arrow[r, "1"] \arrow[d] & \bar{\iota} \arrow[d] \\
\iota \arrow[r] & \rho x
\end{tikzcd}$$

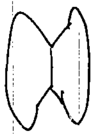
$(\iota, \alpha\alpha_t)$
$\overset{\shortparallel}{\mathbb{C}\lambda^\alpha(t)}$

$\sqrt{d(\rho)}\,\iota(S_k^*)\,\bar{R}_\iota$

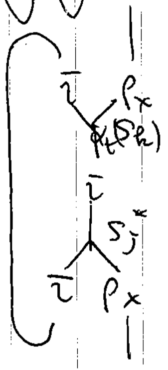
$(\rho_x, \iota\bar{\iota}) = \sqrt{d(\rho)}\,\iota(S_j^*)\,\bar{R}_\iota$



$1 = c^2$

$= c^2$

$= c^2 \frac{1}{d(\iota)} \frac{1}{d(\rho)}$

$= e^2 d(\rho)^{-1}$

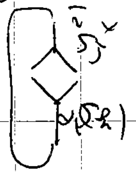


$\sqrt{d(\rho)}\,\bar{R}_\iota^*\,\iota(S_k)\cdot\lambda^\alpha(t)^*\cdot 1\cdot\sqrt{d(\rho)}\,\iota(S_j^*)\bar{R}_\iota$

$= d(\rho)\ \bar{R}_\iota^*\ \iota(S_k)\ \lambda^\alpha(t)^*\ \iota(S_j^*)\ \bar{R}_\iota$

$= d(\rho)\ \bar{R}_\iota^*\ \iota(\alpha_t(S_k))\ \iota(S_j^*)\ \bar{R}_\iota$

$= d(\rho)\ \phi_{\bar{\iota}}\left(\alpha_t(S_k)\,S_j^*\right)$

$= d(\rho_x)\cdot\frac{1}{d(\rho_x)}\cdot\frac{1}{d(\iota)}$


$= S_j^*\,\alpha_t(S_k) = \pi_x(t)_{jk}$

$$\begin{tikzcd}
t \arrow[r] \arrow[d] & \bar{\iota} \arrow[d, "j"] \\
\iota \arrow[r, "k"'] & \rho_x
\end{tikzcd}$$
$= \pi(t)_{jk}$.

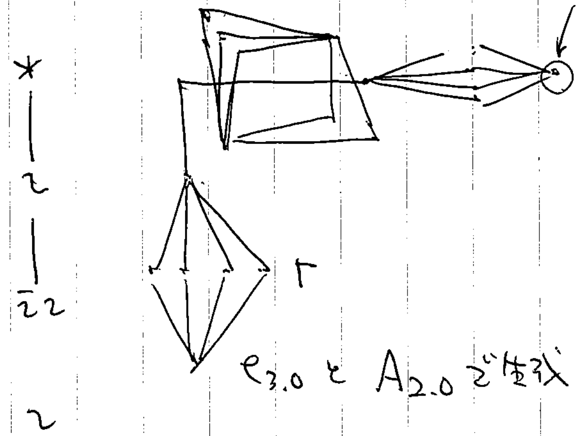
次に、この connection の data が[illegible]に

与えられたとして flat であることを示したい.
直後の計算で.

$$\begin{tikzcd} * \arrow[r, "\rho_+"] \arrow[d, "\tau_+"'] & \cdot & * \arrow[l, "\rho_-"'] \arrow[d, "\tau_-"] \\ \cdot \arrow[r, "\sigma_+"'] & \cdot & \cdot \arrow[l, "\sigma_-"] \end{tikzcd} = \delta_{\tau_+,\tau_-} C_{\rho,\sigma}$$

$\Longleftrightarrow [A_{0,\ell}\ ,\ A_{k,0}] = 0.$

$\ell.\ \ell = |\rho_+|$ $|\tau| = k$



$A_{0.2}$ と $e_{0.3}$ で生成.

$e_{3.0}$ と $A_{2.0}$ で生成

$[A_{2.0}\ ,\ A_{0.2}] = 0$ をしらべる.

$$\begin{tikzcd} \overset{\textcircled{e}}{*} \arrow[r] \arrow[d] & \cdot \arrow[r] & r & \cdot \arrow[l] & x^{e} \arrow[l] \arrow[d] \\ \cdot \arrow[d] & & & & \cdot \arrow[d] \\ s \arrow[r] & \cdot \arrow[r] & t & \cdot \arrow[l] & u \arrow[l] \end{tikzcd}$$

$$= \begin{tikzcd} & \cdot \arrow[r] \arrow[d, "j"] & r \arrow[d] & \cdot \arrow[l] \arrow[d, "k"] & \\ \cdot \arrow[r, "j"'] \arrow[d] & \textcircled{x} \arrow[r] \arrow[d, "\ell"] & \bullet \arrow[d, "k"] & \textcircled{Y} \arrow[l] \arrow[d, "n"] & \cdot \arrow[l, "k"'] \arrow[d] \\ s \arrow[r] & \cdot \arrow[r] & t & \cdot \arrow[l] & u \arrow[l] \end{tikzcd}$$

$$\sum_{p} \begin{tikzcd} \cdot \arrow[r] \arrow[d, "j"'] & r \arrow[d, "p"] & \cdot \arrow[l] \arrow[d, "k"] \\ x \arrow[r, "v"'] & \cdot & Y \arrow[l, "w"] \end{tikzcd} = \sum_{p} \overline{\begin{tikzcd} r \arrow[r] \arrow[d, "p"'] & \cdot \arrow[d, "j"] \\ \cdot \arrow[r, "\tilde{v}"'] & x \end{tikzcd}} \sqrt{\frac{d(x)}{d(v)^2}} \cdot \begin{tikzcd} r \arrow[r] \arrow[d, "p"'] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "\widetilde{w}"'] & Y \end{tikzcd} \sqrt{\frac{d(v)^2}{d(Y)}}$$

$$= \sqrt{\frac{d(x)}{d(Y)}}\ \overline{\pi_x(r)_{j\hat{v}}}\ \pi_{Y}(r)_{k\widetilde{w}}$$

$$\begin{tikzcd} r \arrow[r] \arrow[d, "p"'] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "w"'] & Y \end{tikzcd}$$

$$\overline{\begin{tikzcd} \cdot \arrow[r] \arrow[d, "k"'] & r \arrow[d, "p"] \\ Y \arrow[r, "w"'] & \cdot \end{tikzcd}} = \begin{tikzcd} r \arrow[r] \arrow[d, "p"'] & \cdot \arrow[d, "k"] \\ \cdot \arrow[r, "\widetilde{W}"'] & Y \end{tikzcd} \sqrt{\frac{d(v)^2}{d(Y)}}$$

$$\sum_{i,x}\ \begin{tikzcd}
 & \bullet \arrow[r] \arrow[d, "j"] & r \arrow[d] & \bullet \arrow[l] \arrow[d, "k"] & \\
\bullet \arrow[r, "i"] \arrow[d] & x \arrow[r, "\ell"] \arrow[d, "a"] & \bullet \arrow[d] & Y \arrow[l, "m"] \arrow[d, "b"] & \bullet \arrow[l, "k"] \arrow[d] \\
s \arrow[r] & \bullet \arrow[r] & t & \bullet \arrow[l] & u \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "i"'] & r \arrow[d] \\
x \arrow[r, "\ell"'] & \bullet
\end{tikzcd} = \sqrt{\frac{d(x)}{d(l)^2}}\ \overline{\begin{tikzcd}
r \arrow[r] \arrow[d] & \bullet \arrow[d, "j"] \\
\bullet \arrow[r, "\tilde{\ell}"'] & x
\end{tikzcd}} = \sqrt{\frac{d(x)}{d(l)^2}}\ \overline{\pi_x(r)_{j\tilde{\ell}}}$$

$$\begin{tikzcd}
\bullet \arrow[r, "j"] \arrow[d] & x \arrow[d, "a"] \\
s \arrow[r] & \bullet
\end{tikzcd} = \sqrt{\frac{d(x)}{d(l)^2}}\ \overline{\begin{tikzcd}
s \arrow[r] \arrow[d] & \bullet \arrow[d, "\tilde{a}"] \\
\bullet \arrow[r, "j"'] & x
\end{tikzcd}} = \sqrt{\frac{d(x)}{d(l)^2}}\ \overline{\pi_x(s)_{\tilde{a}j}}$$

$$\begin{tikzcd}
x \arrow[r, "\ell"] \arrow[d, "a"'] & \bullet \arrow[d] \\
\bullet \arrow[r] & t
\end{tikzcd} = \sqrt{\frac{d(l)^2}{d(x)}}\ \overline{\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "\tilde{a}"'] & t \arrow[d] \\
x \arrow[r, "\tilde{\ell}"'] & \bullet
\end{tikzcd}} = \pi_x(t)_{\tilde{a}\tilde{\ell}}$$

$$\sum_{a,j,\ell} \frac{d(x)}{d(l)^2} \cdot \overline{\pi_x(r)_{j\tilde{\ell}}}\,\overline{\pi_x(s)_{\tilde{a}j}}\ \pi_x(t)_{\tilde{a}\tilde{\ell}}$$

$$= \sum_{a,\ell} \frac{d(x)}{d(l)^2}\ \overline{\pi_x(sr)_{\tilde{a}\tilde{\ell}}}\ \pi_x(t)_{\tilde{a}\tilde{\ell}}$$

$$\overline{\pi_x(sr)_{\tilde{a}\tilde{\ell}}} = \pi_x(sr)^*_{\tilde{\ell}\tilde{a}}$$

$$= \sum_{\ell} \frac{d(x)}{d(l)^2} \left(\pi_x(sr)^* \cancel{\pi_x(t)}\right)_{\tilde{\ell}\tilde{\ell}}$$

$$= \frac{d(x)}{d(l)^2} \mathrm{Tr}(\pi_x(r^{-1}s^{-1}t)).$$

$$\frac{d(Y)}{d(l)^2}\ \overline{\mathrm{Tr}(\pi_Y(r^{-1}u^{-1}t))}$$

$$= \frac{d(x)}{|\Gamma|}\ \chi_x(r^{-1}s^{-1}t) \quad \frac{d(Y)}{\cancel{d(l)}|\Gamma|}\ \overline{\chi_Y(r^{-1}u^{-1}t)}\ \cancel{\mathrm{Tr}_Y(t^{-1}ur)}$$

$$\sum_x \frac{d(x)}{|\Gamma|}\ \chi_x(g) = \delta_{g,e}.$$

$$\| $$

$$\sum_x \frac{\overline{\chi_x(e)}\ \chi_x(g)}{|\Gamma|}.$$

$$\xrightarrow{\sum_{x,Y}} \delta_{r^{-1}s^{-1}t,e}\quad \delta_{t^{-1}ur,e}$$

$t = sr.$ $\quad t = ur$

$\rightarrow s = u.$

次に

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \cdot \arrow[r, "a"] \arrow[d] & X \arrow[d, "b"] & \cdot \arrow[l, "c"'] \arrow[d] & 1 \arrow[l] \arrow[d] \\
\cdot \arrow[r] \arrow[d, "i"'] & r \arrow[r] \arrow[d] & \cdot \arrow[d, "l"] & s \arrow[l, Leftrightarrow] \arrow[d] & \cdot \arrow[l] \arrow[d, "q"] \\
Y \arrow[r, "j"'] & \cdot \arrow[r, "k"'] & Z & \cdot \arrow[l, "m"] & W \arrow[l, "p"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \cdot \arrow[r, "a"] \arrow[d] & X \arrow[d, "b"] \\
\cdot \arrow[r] \arrow[d, "i"'] & r \arrow[r] \arrow[d] & \cdot \arrow[d, "l"] \\
Y \arrow[r, "j"'] & \cdot \arrow[r, "k"'] & Z
\end{tikzcd}$$

$$\sqrt{\frac{d(X)d(W)}{d(1)^4}}\cdot \pi_X(s)_{\tilde{b}c}\cdot \pi_W(s)_{q\tilde{p}}\cdot \overline{\pi_Z(s)_{lm}}$$

$$= \sqrt{\frac{d(X)}{d(1)^2}}\ \overline{\pi_X(r)_{\tilde{b}a}}\cdot \sqrt{\frac{d(Y)}{d(1)^2}}\ \overline{\pi_Y(r)_{i\tilde{j}}}\quad \pi_Z(r)_{lk}.$$

$\pi_Z(s^{-1}r)_{mk}$

Σ_l

$$\sum_{\substack{b.l\\ r.s}} \frac{d(X)}{d(1)^4}\sqrt{d(Y)d(W)}\quad \overline{\pi_X(r)_{\tilde{b}a}}\ \overline{\pi_Y(r)_{i\tilde{j}}}\ \pi_Z(r)_{lk}\ \pi_X(s)_{\tilde{b}c}\ \pi_W(s)_{q\tilde{p}}\ \overline{\pi_Z(s)_{lm}}$$

$\pi_X(r^{-1}s)_{ac}$

$$\sum_{r.s} \pi_X(r^{-1}s)_{ac}\ \overline{\pi_Y(r)_{i\tilde{j}}}\quad \pi_Z(s^{-1}r)_{mk}\ \pi_W(s)_{q\tilde{p}}$$

$$s\to rs = \sum_{r.s} \pi_X(s)_{ac}\ \overline{\pi_Y(r)_{i\tilde{j}}}\ \pi_Z(s^{-1})_{mk}\ \pi_W(rs)_{q\tilde{p}}$$

$$\sum_{r} \overline{\pi_Y(r)_{i\hat{j}}}\, \pi_W(rs)_{q\hat{p}}$$

$$\overset{\pi_Y(r)^*}{\overset{\shortparallel}{}}$$

$$= \sum_{r,\alpha} \overline{\pi_Y(r)_{i\hat{j}}}\, \pi_W(r)_{q\alpha}\, \pi_W(s)_{\alpha\hat{p}}$$

$$= \sum_{\alpha} |\Gamma| \cdot h\left(\pi_Y(\bullet)_{i\hat{j}}^{*}\, \pi_W(r)_{q\alpha}\right) \pi_W(s)_{\alpha\hat{p}}$$

$$= \sum_{\alpha} |\Gamma| \cdot \frac{1}{|\Gamma|} \cdot \delta_{iq} \cdot \delta_{\hat{j}\alpha} \cdot \delta_{Y.W}\, \pi_W(s)_{\alpha\hat{p}}$$

$$= \delta_{Y.W}\, \delta_{iq} \cdot \pi_Y(s)_{\hat{j}\hat{p}}$$

$$\begin{matrix}\delta_{Y,W} \\ \delta_{iq}\end{matrix} \quad \sum_{s} \pi_X(s)_{ac}\, \pi_Z(s^{-1})_{mk}\, \pi_Y(s)_{\hat{j}\underset{\uparrow}{\hat{p}}}$$

iq. にかんけいない数.