ERRATA: COMPACT QUANTUM ERGODIC SYSTEMS

REIJI TOMATSU

When $-1 \leq q < 0$, our classification lists of right coideals of $C(SU_q(2))$ [T, p.2, Theorem 7.1, 8.1] are incorrect. Several results have to be corrected.

1. Correction of [T, Section 7]

1.1. A'_∞, D_1, A'_n case. A critical error is the discussion of [T, p.64]. Recall a right coideal $C^*(a,b)$ introduced in [T, p.62], where $a := \sqrt{q_0}x + v$ and $b := \sqrt{q_0}u + y$ with $q_0 := -q > 0$. We labelled it by D_1, but correctly, it is of type A'_∞.

Theorem 1.1. *Let $-1 \leq q < 0$. A right coideal corresponding to the 1st vertex of the A'_∞-graph in* [T, p.82, Fig.15.] *is conjugate to $C^*(a,b)$ by β^L.*

Proof. Let $A := C^*(a,b)$. Then A is linearly spanned by $a^k b^\ell$ with k, ℓ non-negative integers. Using $U_q(su(2))$'s action, we can show that the space of highest weight vectors for each half spin is one-dimensional. This is only the case when A corresponds to the vertex with a loop in the A'_∞-graph. Their uniqueness up to the conjugation by β^L is discussed in [T, p.62]. □

So, we must again study whether D_1 and A'_n types are allowed or not.

Lemma 1.2. *Let $-1 < q < 0$. A right coideal with $\pi_{1/2}$-multiplicity 1 is conjugate to $C^*(a,b)$ by β^L. In particular, types D_1 and A'_n $(3 \leq n < \infty)$ do not appear.*

Proof. Let A be such a right coideal. We may and do assume that $C^*(a,b) \subset A$ using β^L. Recall the product map Ψ defined in [T, p.34]. By direct computation, we can show the following claim.

Claim 1. Let $\nu \in (1/2)\mathbb{Z}_+$ and $\eta := \sum_{t \in I_\nu} d_t \boldsymbol{w}_t^\nu$. Then $\Psi_{\nu-1/2}((a,b),\eta) = 0$ if and only if d_t satisfies the following recurrence formula:

$$(-1)^{\nu-t}\, q_0^{\nu-t+1/2}\sqrt{1-q_0^{2(\nu+t+1)}}\,d_{t+1} = \sqrt{1-q_0^{2(\nu-t)}}\,d_t \quad t \in I_\nu.$$

Its exact solution is given by the following:

$$d_t := (-1)^{(\nu+t)(5\nu+t-1)/2}\, q_0^{(\nu+t)(-3\nu+t-2)/2} \begin{bmatrix} 2\nu \\ \nu+t \end{bmatrix}_{q^2}^{1/2} d_{-\nu}.$$

On the conjugation of η, we have $T\eta = \sum_{t \in I_\nu} e_t \boldsymbol{w}_t^\nu$ and $e_t = (-1)^{(6\nu-1)t} q_0^{(2\nu+1)t} d_t$. For $n \in \mathbb{N}$, we have

$$a^n = \sum_{t \in I_\nu} f_t^n w(\pi(n/2))_{t,-n/2}, \quad f_t^n := q_0^{-t/2+n/4} \begin{bmatrix} n \\ t+n/2 \end{bmatrix}_q \begin{bmatrix} n \\ t+n/2 \end{bmatrix}_{q^2}^{-1/2},$$

Thus we have three linearly independent π_ν-eigenvectors $\eta, T\eta$ and $\sum_{t \in I_\nu} f_t^{2\nu} \boldsymbol{w}_t^\nu$.

Now assume that A is of type D_1. Then its spectral pattern is like:

$$\pi_0 \oplus \pi_{1/2} \oplus \pi_1 \oplus 2\pi_{3/2} \oplus \cdots . \tag{1.1}$$

Applying the above claim to $\nu = 3/2$, we get a contradiction.

If A is a right coideal corresponding to the end vertex of the A'_n-graph with $3 \leq n < \infty$, then its spectral pattern corresponding to the end vertex is like:

$$\pi_0 \oplus \pi_1 \oplus \cdots \oplus \pi_{1/2} \oplus \pi_{3/2} \oplus \cdots \oplus \pi_{n-3/2} \oplus 2\pi_{n-1/2} \oplus 3\pi_n \cdots . \tag{1.2}$$

So, putting $\nu = n - 1/2$ in the above, we get a contradiction again. $\square$

By [T, Proposition 4.22], there really exists a right coideal corresponding to each vertex in the A'_∞-graph in [T, p.82 Fig.15]. In this note, we will say that a right coideal is of type $A'_{n,k}$ when it is of type A'_n and corresponding to the k-th vertex, which is verified by computing the spectral multiplicities.

Set $m := k - 1/2$ and define the following π_m-eigenvector:

$$\zeta^{m,+} = \sum_{t \in I_m} q_0^{-(m+t)(2m-2t+1)/2} (i^{m-t} + i(-i)^{m-t}) \begin{bmatrix} 2m \\ m-t \end{bmatrix}_{q^2}^{1/2} \boldsymbol{w}_t^m.$$

Theorem 1.3. *A right coideal of type $A'_{\infty,k}$ is conjugate to $C^*(\zeta^{k-1/2,+})$ by β^L.*

Proof. Let A be such a right coideal. Then its spectral pattern is:

$$\bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \pi_\ell \oplus \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} \pi_{k-1/2+\ell}. \tag{1.3}$$

Put $m := k - 1/2$. Since the π-multiplicity of A equals 1, we may assume that ξ_λ^1 belongs to A by using β^L, where ξ_λ^1 is the π_1-eigenvector defined in [T, p.55]. Then a π_m-eigenvector of A, $\eta := \sum_{t \in I_m} d_t \boldsymbol{w}_t^m$ satisfies the recurrence formula given in [T, Lemma 7.4]. Then as in the discussion given in [T, p.54], we can deduce $0 < \lambda_0 \leq 1$.

When $0 < \lambda_0 < 1$, we can solve the equation by using $T\eta$ as in [T, p.57]. Indeed its (unique) solution is $\lambda_0 = (q_0^{-n} - q_0^n)(q_0^{-n-1} + q_0^{n+1})^{-1/2}(q_0^{-n+1} + q_0^{n-1})^{-1/2}$, and η is a scalar multiple of $\zeta^{m,+}$. Hence $B := C^*(\zeta^{m,+}) \subset A$. Comparing (1.3) with other spectral patterns, we see B must be of type $A'_{\infty,k}$, and $A = B$.

Next suppose that $\lambda_0 = 1$. Then $\eta = \alpha \boldsymbol{w}_{-m}^m + \beta \boldsymbol{w}_m^m$ for some $\alpha, \beta \in \mathbb{C}$. Since the π_m-eigenvector space is one-dimensional, we can deduce that $\eta^m = q_0^{m/2} \boldsymbol{w}_{-m}^m + q_0^{-m/2} \boldsymbol{w}_m^m$ by using $T\eta^m$ and a conjugation by β^L. Then $(\eta_{-m}^m)^2 = (q_0^{m/2} x^{2m} + q_0^{-m/2} v^{2m})^2 = q_0^m x^{4m} + (1 - q_0^{4m^2}) x^{2m} v^{2m} + q_0^m v^{4m}$ that is linearly independent from $(\boldsymbol{w}_0^1)^{2m} = (1 + q_0^2)^m (xv)^{2m}$. This contradicts (1.3) at π_m. $\square$

The discussion about the types $\mathbb{T}_n, D_n$ (odd $n \geq 3$) in [T, p.56–59] is correctly working. The following result corrects [T, Theorem 7.1].

Theorem 1.4. *Let $-1 < q < 0$. Then the possible types of right coideals of $C(SU_{-1}(2))$ are one of $1, \mathbb{T}_n$ $(n \geq 2), \mathbb{T}, SU(2), D_\infty^*$ and $A'_{\infty,k}$ $(k \geq 1)$ listed in* [T, p.79–82]. *A right coideal of type $\mathbb{T}$ is one of the quantum spheres. Otherwise, they are unique up to conjugation by β^L. In particular, a type $A'_{\infty,k}$ right coideal is conjugate to $C^*(\zeta^{k-1/2,+})$ by β^L.*

2. Correction of [T, Section 8]

We next study when $q = -1$.

2.1. D_1 **case.** Theorem 1.1 shows $a := x + v$ and $b := u + y$ generate a right coideal of type $A'_{\infty,1}$.

Theorem 2.1. *Let $A \subset C(SU_{-1}(2))$ be a right coideal of type D_1. Then A is equal to $C^*(a, b, \zeta^{3/2})$ up to conjugation by β^L, where*

$$\zeta^{3/2} = \boldsymbol{w}^{3/2}_{-3/2} - \sqrt{3}\boldsymbol{w}^{3/2}_{-1/2} - \sqrt{3}\boldsymbol{w}^{3/2}_{1/2} + \boldsymbol{w}^{3/2}_{3/2}.$$

This algebra indeed coincides with $C(D_1\backslash SU_{-1}(2))$

Proof. Note that such A really exists because of the existence of the subgroup D_1 of $SU_{-1}(2)$. By (1.1), the linear operator $\Psi_{1/2}((a, b), \cdot)$ from the $\pi_{3/2}$-eigenvector space of A into the $\pi_{1/2}$-eigenvector space of A has one-dimensional kernel that is spanned by $\zeta^{3/2}$ by Claim 1 in Lemma 1.2 with $q_0 = 1$. So, A must contain $B := C^*(a, b, \zeta^{3/2})$ that is never of type A'_n by (1.2) and $3 \leq n \leq \infty$. Hence B is of type D_1 and $A = B$.

Next we check that $B = C(D_1\backslash SU_{-1}(2))$. Let $D_1 := \{e, g\} \cong \mathbb{Z}_2$. We use the following embedding D_1 into $SU_{-1}(2)$: $x(g) = 0 = y(g)$, $u(g) = 1 = v(g)$. It is straightforward to check that $a, b, \zeta^{3/2}_t$ are belonging to $C(D_1\backslash SU_{-1}(2))$. □

2.2. A'_∞ **case.** Let A be a right coideal of type $A'_{\infty,k}$. Applying the discussion preceding [T, Lemma 8.3] to $m := k-1/2$ (see (1.3)), we can show a π_1-eigenvector is obtained by $\boldsymbol{w}^1_0$ or $\xi^{\pi/2,\pi/2} := \boldsymbol{w}^1_{-1} + \boldsymbol{w}^1_1$ though its definition here is slightly different from the one in [T, Lemma 8.2]. If $\boldsymbol{w}^1_0$ belongs to A, a π_m-eigenvector is a linear combination of $\boldsymbol{w}^m_{-m}$ and $\boldsymbol{w}^m_m$. By using β^L and the conjugation T, we may assume that $\boldsymbol{w}^m_{-m} + \boldsymbol{w}^m_m$ belongs to A, but $C^*(\boldsymbol{w}^m_{-m} + \boldsymbol{w}^m_m)$ is $C(D_{2m}\backslash SU_{-1}(2))$ (see [T, p.75]), and this is a contradiction. Hence $\xi^{\pi/2,\pi/2}$ belongs to A, and by [T, Lemma 8.3] with $\chi = \pi/2$, a π_m-eigenvector is a linear combination of

$$\eta^m := \sum_{t\in I_m} i^{m-t} \begin{bmatrix} 2m \\ m-t \end{bmatrix}^{1/2} \boldsymbol{w}^m_t, \quad \hat{\eta}^m := \sum_{t\in I_m} (-i)^{m-t} \begin{bmatrix} 2m \\ m-t \end{bmatrix}^{1/2} \boldsymbol{w}^m_t.$$

By direct computation, we have the following lemma.

Lemma 2.2. *One has*

$$\Psi_m(\xi^{\pi/2,\pi/2}, \eta^m) = \sqrt{2}i\hat{\eta}^m, \quad \Psi_m(\xi^{\pi/2,\pi/2}, \hat{\eta}^m) = -\sqrt{2}i\eta^m.$$

Hence to get the multiplicity 1 at the spin m, the eigenvector must be a scalar multiple of $\eta^m \pm i\hat{\eta}^m$. Let us prepare the following vectors:

$$\zeta^{\nu,+} := \eta^\nu + i\hat{\eta}^\nu, \quad \zeta^{\nu,-} := \eta^\nu - i\hat{\eta}^\nu, \quad \nu \in (1/2)\mathbb{Z}_{\geq 0}.$$

Then we can prove the following lemmas.

Lemma 2.3. *For all $\nu \in (1/2)\mathbb{Z}_{\geq 0}$, one has $\beta^L_i(\zeta^{\nu,\pm}) = \pm i^{-2\nu+1}\zeta^{\nu,\mp}$.*

Lemma 2.4. *One has*

$$T\zeta^{\nu,\pm} = \begin{cases} i^{-2\nu}\zeta^{\nu,\mp} & \text{if } \nu \in \mathbb{Z}_{\geq 0}, \\ i^{-2\nu}\zeta^{\nu,\pm} & \text{if } \nu \in 1/2 + \mathbb{Z}_{\geq 0}. \end{cases}$$

Lemma 2.5. *For all $\nu \in (1/2)\mathbb{Z}_{\geq 1}$, one has*

$$\Psi_{\nu-1/2}((a,b),\zeta^{\nu,-}) = 0, \quad \Psi_{\nu+1/2}((a,b),\zeta^{\nu,-}) = -i\zeta^{\nu+1/2,+}.$$

Denote by $X_\nu^\pm$ the linear span of $\zeta_t^{\nu,\pm}$ for all $t \in I_\nu$. The above lemmas imply that the following useful formulae:

$$(X_\nu^\pm)^* = \begin{cases} X_\nu^\mp & \text{if } \nu \in \mathbb{Z}_{\geq 0}, \\ X_\nu^\pm & \text{if } \nu \in 1/2 + \mathbb{Z}_{\geq 0}, \end{cases} \tag{2.1}$$

$$\beta_i^L(X_\nu^\pm) = X_\nu^\mp, \quad X_{1/2}^+ X_\nu^- = X_{\nu+1/2}^+ \quad \text{for all } \nu \in (1/2)\mathbb{Z}_{\geq 0}. \tag{2.2}$$

Also we have $\zeta_{-\nu}^{\nu,+} = i(x+v)\zeta_{-\nu+1/2}^{\nu-1/2,-}$, and by Lemma 2.5,

$$\zeta_{-\nu}^{\nu,+} = \begin{cases} i^{2\nu}(1-i)[(x+v)(x-v)]^\nu & \text{if } \nu \in \mathbb{Z}_{\geq 0}, \\ i^{2\nu}(1-i)[(x+v)(x-v)]^{\nu-1/2}(x+v) & \text{if } \nu \in 1/2 + \mathbb{Z}_{\geq 0} \end{cases} \tag{2.3}$$

Theorem 2.6. *The right coideal of type $A'_{\infty,k}$ is conjugate to $C^*(\zeta^{k-1/2,+})$ by β^L.*

Proof. Let A be a right coideal of type $A'_{\infty,k}$, which really exists. By Lemma 2.3, we may assume A contains $C^*(\zeta^{k-1/2,+})$. Then $C^*(\zeta^{k-1/2,+})$ must be of type $A'_{\infty,k}$, and $A = C^*(\zeta^{k-1/2,+})$. □

Fix $m \in 1/2+\mathbb{Z}_{\geq 0}$. For $\nu \in 1/2+\mathbb{Z}_{\geq 0}$, we will introduce the π_ν-eigenvector $\xi^{\nu,\pm}$ whose highest weight vector $\xi_{-\nu}^{\nu,\pm}$ equals $(x^2+v^2)^{\nu-m}\zeta_{-m}^{m,\pm}$. Note that $\xi^{\nu,\pm}$ belong to $C^*(\zeta^{m,\pm})$ and they densely span it for $\nu \geq m$. Recall $\xi^{\pi/2,\pi/2} = \boldsymbol{w}_{-1}^1 + \boldsymbol{w}_1^1$ in what follows.

Lemma 2.7. *For any $\ell \in \mathbb{Z}_{\geq 1}$, one has a non-zero $c_{m,\ell}^\pm \in \mathbb{C}$ such that $\Psi_{m+\ell-1}(\xi^{\pi/2,\pi/2}, \xi^{m+\ell,\pm}) = c_{m,\ell}^\pm \xi^{m+\ell-1,\pm}$.*

Proof. By Lemma 2.3, it suffices to show the statement for $\xi^{m+\ell,+}$. Suppose that we would have $\Psi_{m+\ell-1}(\xi^{\pi/2,\pi/2}, \xi^{m+\ell,+}) = 0$ for some $\ell \geq 1$. Set $\nu := m+\ell$. Then ξ^ν is a linear combination of η^ν and $\hat{\eta}^\nu$ by [T, Lemma 8.3]. Since $\xi^{\pi/2,\pi/2}$ and ξ^ν belong to $C^*(\zeta^{m,+})$ and its π_ν-multiplicity equals 1, we see $\xi^\nu = c\zeta^{\nu,+}$ or $\xi^\nu = c\zeta^{\nu,-}$ for some $c \in \mathbb{C}$ by Lemma 2.2. If $\xi^\nu = c\zeta^{\nu,+}$, by using (2.3) and changing c, we have

$$(x^2+v^2)^\ell[(x+v)(x-v)]^{m-1/2}(x+v) = c[(x+v)(x-v)]^{\nu-1/2}(x+v).$$

Applying the counit, we get $c = 1$. Put $r := m - 1/2$. Multiplying $x+v$ from the right, we get

$$(x^2+v^2)^{\ell+1} \cdot [(x+v)(x-v)]^r = (x^2+v^2)[(x+v)(x-v)]^{r+\ell}.$$

Using the map υ_g defined in [T, p.31] with $g = r^{23}(\theta)$, we obtain

$$\upsilon_g(x^2+v^2) = \cos 2\theta, \quad \upsilon_g((x+v)(x-v)) = 1 - \sin\theta\sigma_3 = \begin{pmatrix} 1-\sin\theta & 0 \\ 0 & 1+\sin\theta \end{pmatrix}.$$

Hence the above equality shows $\cos^{\ell+1}(2\theta)(1-\sin\theta)^r = \cos 2\theta(1-\sin\theta)^{r+\ell}$. If we put $\theta = 3\pi/2$, we get $(-1)^{\ell+1} = -2^\ell$, that is, $\ell = 0$. This is a contradiction.

If $\xi^\nu = c\zeta^{\nu,-}$, by Lemma 2.3, (2.3) and changing c, then we have

$$(x^2+v^2)^\ell[(x+v)(x-v)]^r(x+v) = c[(x-v)(x+v)]^{r+\ell}(x-v).$$

Hence $c = 1$, and multiplying $x+v$ from the right, we have

$$(x^2+v^2)^{\ell+1}[(x+v)(x-v)]^r = [(x-v)(x+v)]^{r+\ell+1}.$$

Since $\upsilon_g((x-v)(x+v)) = 1+\sin\theta\sigma_3$, we get $\cos^{\ell+1}(2\theta)(1-\sin\theta)^r = \cos 2\theta(1+\sin\theta)^{r+\ell+1}$. Put $\theta = 3\pi/2$ and we get $(-1)^{\ell+1}2^r = 0$, a contradiction.

Hence $\Psi_{m+\ell-1}(\xi^{\pi/2,\pi/2}, \xi^{m+\ell,+})$ is a non-zero $\pi_{m+\ell-1}$-eigenvector of $C^*(\zeta^{m,+})$. Since its $\pi_{m+\ell-1}$-eigenvector space is one-dimensional, that is a scalar multiple of $\xi^{m+\ell-1,+}$. □

2.3. A'_ℓ ($\ell \geq 3$) **case.**

Lemma 2.8. *One has the following statements:*

(1) *If $k \in 1/2 + \mathbb{Z}_{\geq 0}$ and $\ell \in (1/2)\mathbb{Z}_{\geq 0}$, then $X_k^+ X_\ell^- = X_{k+\ell}^+$;*
(2) *If $k \in 1/2 + \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{\geq 0}$, then $X_\ell^+ X_k^+ = X_{k+\ell}^+$ and $X_\ell^- X_k^- = X_{k+\ell}^-$.*

Proof. (1). By (2.1), we have

$$X_k^+ = X_{1/2}^+ X_{k-1/2}^- = X_{1/2}^+\beta_i(X_{k-1/2}^+) = X_{1/2}^+\beta_i(X_{1/2}^+ X_{k-1}^-) = X_{1/2}^+\beta_i(X_{1/2}^+)X_{k-1}^+.$$

This implies $X_k^+ = (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-1/2}X_{1/2}^+$. Hence

$$\begin{aligned} X_k^+ X_\ell^- &= (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-1/2}X_{1/2}^+X_\ell^- = (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-1/2}X_{\ell+1/2}^+ \\ &= (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-3/2}X_{1/2}^+\beta_i(X_{1/2}^+X_{\ell+1/2}^-) \\ &= (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-3/2}X_{1/2}^+\beta_i(X_{\ell+1}^+) \\ &= (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-3/2}X_{1/2}^+X_{\ell+1}^- \\ &= (X_{1/2}^+\beta_i(X_{1/2}^+))^{k-3/2}X_{\ell+3/2}^+. \end{aligned}$$

We can inductively show that this equals to $X_{k+\ell}^+$.

(2). Use (2.1) and (2.2). □

Lemma 2.9. *Let $m, n \in 1/2 + \mathbb{Z}_{\geq 0}$ and $m \leq n$. One has the following equalities:*

(1) $(X_m^+X_n^-)^\ell X_m^+ = X_{\ell(m+n)+m}^+$;
(2) $(X_m^+X_n^-)^\ell = X_{\ell(m+n)}^+$. *In particular,* $(X_{m+n}^+)^\ell = X_{\ell(m+n)}^+$;
(3) $(X_n^-X_m^+)^\ell X_n^- = X_{\ell(m+n)+n}^-$;
(4) $(X_n^-X_m^+)^\ell = X_{\ell(m+n)}^-$.

Proof. (1). The previous lemma implies $(X_m^+X_n^-)^\ell X_m^+ = (X_{m+n}^+)^\ell X_m^+$ and $X_{m+n}^+X_m^+ = X_{2m+n}^+$ since $m+n \in \mathbb{Z}$. Similarly, $(X_{m+n}^+)^\ell X_m^+ = X_{\ell(m+n)+m}^+$.

(2). First, we have

$$X_m^+X_n^-X_m^+ = X_m^+(X_m^+X_n^-)^* = X_m^+(X_{m+n}^+)^* = X_m^+X_{m+n}^- = X_{2m+n}^+.$$

Since $2m+n \in 1/2+\mathbb{Z}_{\geq 0}$, $(X_m^+X_n^-)^2 = X_{2m+n}^+X_n^- = X_{2(m+n)}^+$.

Next if $(X_m^+X_n^-)^{\ell-1} = X_{(\ell-1)(m+n)}^+$, then $(X_m^+X_n^-)^{\ell-1}X_m^+ = X_{(\ell-1)(m+n)+m}^+$, and $(X_m^+X_n^-)^\ell = X_{\ell(m+n)}^+$. By induction, we are done.

(3), (4). Use (2.2) in (1) and (2). $\square$

Set $C_m^\pm := C^*(X_m^\pm) = C^*(\zeta^{m,\pm})$ for $m \in (1/2)\mathbb{Z}_{\geq 0}$. When $m \in 1/2+\mathbb{Z}_{\geq 0}$, we have shown $C_m^\pm$ is of type $A'_{\infty,m+1/2}$ in Theorem 2.6. Thus if we put $Q := C(S^2_{-1,0})$, then $C_m^\pm = Q + QX_m^\pm = Q + X_m^\pm Q$. It is useful to note $QX_m^\pm = X_m^\pm Q$, that is, we can treat Q as a "coefficient algebra"" in what follows.

Lemma 2.10. *Let $A_{m,n} := C^*(\zeta^{m,+},\zeta^{n,-}) = C^*(X_m^+, X_n^-)$ for $m,n \in 1/2+\mathbb{Z}_{\geq 0}$ and $m \leq n$. If $1/2 < m < n$, then $A_{m,n}$ is of type $A'_{m+n,m+1/2}$.*

Proof. Let us first assume $m \leq n$. Since $X_m^+X_m^+, X_n^-X_n^- \subset Q$, $A_{m,n}$ is the norm closure of

$$Q + \sum_{\ell\in\mathbb{Z}_{\geq 0}} Q(X_m^+X_n^-)^\ell X_m^+ + Q(X_m^+X_n^-)^\ell + Q(X_n^-X_m^+)^\ell + Q(X_n^-X_m^+)^\ell X_n^-,$$

which is, by the previous lemma, simply written as follows:

$$Q + \sum_{\ell\in\mathbb{Z}_{\geq 0}} QX_{\ell(m+n)+m}^+ + QX_{\ell(m+n)}^+ + QX_{\ell(m+n)}^- + QX_{\ell(m+n)+n}^-.$$

Hence $A_{m,n}$ $(m \leq n)$ is the norm closure of

$$C^*(X_{m+n}^+, X_{m+n}^-) + \sum_{p\in\mathbb{Z}_{\geq 0}} C_{p(m+n)+m}^+ + C_{p(m+n)+n}^-. \tag{2.4}$$

because $X_{\ell(m+n)}^+ = (X_{m+n}^+)^\ell$ by Lemma 2.9 (2). Note that $(X_{m+n}^+)^* = X_{m+n}^-$.

Claim 2. For $p \in \mathbb{Z}_{\geq 1}$, $C^*(X_p^+, X_p^-)$ contains Q, and is of type $\mathbb{T}_{2p}$.

Proof of Claim. The map υ_g introduced in [T, Proof of Proposition 8.11] satisfies

$\rho_g(x^{2p}+v^{2p}) = \sqrt{2}^{-2p} \otimes (\eta_{-p}^{p,0} + \hat{\eta}_{-p}^{p,0})$, $\rho_g(x^{2p}-v^{2p}) = -i\sqrt{2}^{-2p}\sigma_3 \otimes (\eta_{-p}^{p,0} - \hat{\eta}_{-p}^{p,0})$,

where $\eta^{p,0}$ and $\hat{\eta}^{p,0}$ are the π_p-eigenvectors defined in [T, p.67]. Take a character $C(\sigma_3)^* \to \mathbb{C}$, we obtain an $SU_{-1}(2)$-isomorphism from $C^*(\boldsymbol{w}_{-p}^p, \boldsymbol{w}_p^p)$ onto $C^*(\eta_{-p}^{p,0}, \hat{\eta}_{-p}^{p,0})$. Since $\eta^p = e^{im\pi/2}\beta_{e^{i\pi/4}}^L(\eta^{p,0})$ and $\eta^p = e^{im\pi/2}\beta_{e^{i\pi/4}}^L(\hat{\eta}^{p,0})$, we get an $SU_{-1}(2)$-isomorphism from $C^*(\boldsymbol{w}_{-p}^p, \boldsymbol{w}_p^p) = C(\mathbb{T}_{2p}\backslash SU_{-1}(2))$ onto $C^*(X_p^+, X_p^-)$. Since $\rho_g(x^2+v^2) = 1 \otimes (x^2+v^2)$, the isomorphism maps Q onto itself. $\square$

So, if $m < n$, then its spectral pattern is like:

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} (1+2[k/(m+n)])\pi_k \oplus \bigoplus_{m\leq k\leq n-1, k\in 1/2+\mathbb{Z}_{\geq 0}} \pi_k \oplus 2\pi_n \oplus \cdots .$$

Since $m > 1/2$, $A_{m,n}$ is not of type D_1, and this corresponds to the $(m+1/2)$-th vertex of the A'_{m+n}-graph. $\square$

We have proved the existence of a right coideal of type $A'_{\ell,k}$ for any $\ell \geq 3$ and $k \leq \ell/2$. However, the above $C^*(\zeta^{m,+}, \zeta^{n,-})$ when $m < n$ does not realize a right coideal corresponding to the middle vertex of each A'_ℓ-graph for odd $\ell \geq 3$. Let us compute the spectral pattern of $A_{m,m}$. Recall the spectral pattern of $C^{\pm}_{p(m+n)+m}$ in (2.4) with $m = n$ is given by

$$\bigoplus_{k\in\mathbb{Z}_{\geq 0}} \pi_k \oplus \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \pi_{(2p+1)m+k}.$$

From (2.4), for all $\nu \in 1/2 + \mathbb{Z}_{\geq 0}$ with $(2p+1)m \leq \nu < (2p+3)m$, the π_ν-eigenvector space of $A_{m,m}$ comes from the linear span of π_ν-eigenvector spaces of $C^{\pm}_{qm}$ for $q = 1, 3, \dots, 2p+1$, which are all one-dimensional. We will prove they are linearly independent. Let us take a non-zero π_ν-eigenvector $v^{\nu,\pm}_q$ of $C^{\pm}_{qm}$. Suppose that we have

$$\sum_{q=1,3,\dots,2p+1} \alpha^+_q v^{\nu,+}_q + \alpha^-_q v^{\nu,-}_q = 0 \tag{2.5}$$

for some $\alpha^{\pm}_q \in \mathbb{C}$. We have proved $\Psi_{\mu-1}(\xi^{\pi/2,\pi/2}, v^{\mu,\pm}_q) = c_{q,\mu} v^{\mu-1,\pm}_q$ for some non-zero $c_{q,\mu}$ for all $\mu > qm$ in Lemma 2.7. We successively operate the linear maps $\Psi_{\mu-1}(\xi^{\pi/2,\pi/2}, \cdot)$ with $\mu = \nu, \nu-1, \dots, m+1$, and we obtain $c'\alpha^+_1\zeta^{m,+} + c''\alpha^-_1\zeta^{m,-} = 0$ for some non-zero $c', c'' \in \mathbb{C}$. Thus $\alpha^+_1 = 0 = \alpha^-_1$. Similarly, all $\alpha^{\pm}_q$ equal 0.

Thus the right coideal $C^*(\zeta^{m,+}, \zeta^{m,-}) = C^*(\eta^m, \hat{\eta}^m)$ has the exactly same spectral pattern as that of type $\mathbb{T}_{2m}$, that is,

$$\bigoplus_{s\in\mathbb{Z}_{\geq 0}} (1 + 2[s/2m])\pi_s \oplus \bigoplus_{s\in\mathbb{Z}_{\geq 0}} 2[(2s+2m+1)/4m]\pi_{s+1/2}.$$

Theorem 2.11. *Let A be a right coideal of type $A'_{\ell,k}$ with $3 \leq \ell < \infty$ and $k \leq \ell/2 + 1/2$. Then the following statements hold:*

(1) *If $k \leq \ell/2$, then A is conjugate to $C^*(\zeta^{k-1/2,+}, \zeta^{\ell-k+1/2,-})$ by β^L;*
(2) *If ℓ is odd and $k = \ell/2 + 1/2$, then A is conjugate to $C^*(\eta^{\ell/2}, \hat{\eta}^{\ell/2})$ by β^L.*

Proof. (1). Put $m := k - 1/2$ and $n := \ell - k + 1/2$. Since the vertex is not sitting at the central position, the multiplicities of π_1, π_m are equal to 1 and that of π_n is 2. By the same discussion as that of the beginning of the subsection 2.2, a π_1-eigenvector is a scalar multiple of $\boldsymbol{w}^1_0$ or $\xi^{\pi/2,\pi/2}$. If $\boldsymbol{w}^1_0$ would belong to A, then we may assume that A has a π_m-eigenvector $\boldsymbol{w}^m_{-m} + \boldsymbol{w}^m_m$. However, they satisfy $\Psi_m(\boldsymbol{w}^1_0, \boldsymbol{w}^m_{\pm m}) = \mp\boldsymbol{w}^m_{\pm m}$, and $\boldsymbol{w}^m_{-m} - \boldsymbol{w}^m_m$ also belong to A, which shows that the π_m-multiplicity is greater than 1, and this is a contradiction.

Thus the π_1-eigenspace generates the Podleś sphere $C(S^2_{-1,0}) = C^*(\xi^{\pi/2,\pi/2})$, and we may assume that the π_m-eigenvector $\zeta^{m,+}$ belongs to A as in the subsection 2.2. By [T, Lemma 8.3] with $\chi = \pi/2$, we have a non-zero π_n-eigenvector of A that is a linear combination of η^n and $\hat{\eta}^n$. Lemma 2.2 shows if $\zeta^{n,\pm}$ would not belong to A, then we would have three linearly independent π_n-eigenvectors ξ^n, which is the one defined in the preceding paragraph of Lemma 2.7, η^n and $\hat{\eta}^n$

by Lemma 2.2, 2.7. This is a contradiction. Hence there are two possibilities for π_n-eigenvectors, that is, either $\zeta^{n,+}$ or $\zeta^{n,-}$ belong to A. If $\zeta^{n,-}$ does, we are done by Lemma 2.10.

Suppose on the contrary that $\zeta^{n,+}$ belongs to A. Using the Clebsh-Gordan coefficients, we see $\varepsilon(\Psi_{n-1/2}(\zeta^{m,+},\zeta^{n,+})_{-n+1/2}) \neq 0$, and $\Psi_{n-1/2}(\zeta^{m,+},\zeta^{n,+})$ is a nonzero $\pi_{n-1/2}$-eigenvector affiliated with $C(S^2_{-1,0})$ because the $\pi_{n-1/2}$-multiplicity of A equals 1, and the $\pi_{n-1/2}$-eigenspace coincides with that of $C(S^2_{-1,0})$.

Now we use the map $\rho_\theta := \rho_{g^{\theta,\pi/2}}$ defined in [T, p.68]. Then it is easy to see that $\rho_\theta(x) = 1 \otimes x$ for all $x \in C(S^2_{-1,0})$. However, using [T, Lemma 8.4], we obtain $\rho_\theta(\zeta_r^{m,+}\zeta_s^{n,+}) = (\cos(m-n)\theta - i\sin(m-n)\theta\sigma_3) \otimes \zeta_r^{m,+}\zeta_s^{n,+}$ for any $r \in I_m$, $s \in I_n$. Thus $\rho_\theta(\Psi_{n-1/2}(\zeta^{m,+},\zeta^{n,+})) \neq 1 \otimes \Psi_{n-1/2}(\zeta^{m,+},\zeta^{n,+})$, which is a contradiction.

(2). Note that a right coideal of type $A'_{2m,1}$ exists by Lemma 2.10, and so really does a right coideal of type $A'_{2m,m+1/2}$ by [T, Proposition 4.22]. We can check that the π_1-multiplicity equals 1, π_ν-multiplicity 0 for $\nu = 1/2, 3/2, \dots, m-1$ and the π_m-multiplicity 2 from the A'_{2m}-graph. Hence A must contain $C^*(\eta^m, \hat{\eta}^m)$ or $C(\mathbb{T}_{2m}\backslash SU_{-1}(2))$ up to conjugation by β^L from the discussion of [T, p.66–67].

The intertwining matrix argument of [T, p.64], which has not worked correctly there because that $A_{\pi_{1/2}}$ generates a right coideal of type not D_1 but $A'_{\infty,1}$, shows we do not have any equivariant embedding of a right coideal B of type $\mathbb{T}_{2m}$ into A. Indeed, if Λ is a $2m \times 2m$ matrix with non-negative integers such that a column of Λ is of the form ${}^t(0,\dots,0,1,0,\dots,0)$ and $\Lambda\mathbb{M}^B(\pi_{1/2}) = \mathbb{M}^A(\pi_{1/2})\Lambda$, then Λ must be a permutation unitary because the symmetric matrices $\mathbb{M}^A(\pi_{1/2})$ and $\mathbb{M}^B(\pi_{1/2})$ have the common Perron-Frobenius vector ${}^t(1,1,\dots,1)$. However, $\mathrm{Tr}(\mathbb{M}^A(\pi_{1/2})) = 2$ and $\mathrm{Tr}(\mathbb{M}^B(\pi_{1/2})) = 0$, and this is a contradiction.

This argument also implies that $A_{m,m} = C^*(\eta^m, \hat{\eta}^m) \subset A$ and $A_{m,m}$ is not of type $\mathbb{T}_{2m}$. The only possible type of $A_{m,m}$ is $A'_{r,s}$ for some $3 \leq r < \infty$ and $s \leq (r+1)/2$. It is easy to deduce $r = 2m = \ell$ and $s = \ell/2 + 1/2 = k$. □

2.4. $\mathbb{T}_n$ **(n even, odd) case.**

Theorem 2.12. *Let $n \in \mathbb{N}$. Then any right coideal of type $\mathbb{T}_n$ is conjugate to $C(\mathbb{T}_n\backslash SU_{-1}(2))$ by β^L.*

Proof. If n is even, there is nothing to prove. Suppose n is odd. By the argument in [T, p.67], such a right coideal must be $C(\mathbb{T}_n\backslash SU_{-1}(2))$ or contain $C^*(\eta^{n/2}, \hat{\eta}^{n/2})$. We have seen the latter one is of type $A'_{n,(n+1)/2}$ and its spectral pattern is same as that of $C(\mathbb{T}_n\backslash SU_{-1}(2))$. Hence we are done. □

The discussion about type D_n (odd $n \geq 3$) [T, p.73–79] works correctly. The following result is a correction of [T, Theorem 8.1].

Theorem 2.13. *One has the following:*

1. *For $C(SU_{-1}(2))$, all graphs listed in* [T, p.79–82] *can appear;*
2. *A right coideal not of type D_n (odd $n \geq 3$) or type A'_ℓ ($3 \leq \ell \leq \infty$) is a quotient by a closed subgroup of $SU_{-1}(2)$;*
3. *A right coideal of type D_n (odd $n \geq 3$) is conjugated to $C(D_n\backslash SU_{-1}(2))$ or $C^*(\eta^{\frac{n}{2}})$ by β^L. (They are $SU_{-1}(2)$-isomorphic by* [T, Proposition 8.11].)

(4) *A right coideal of type* $A'_{\infty,k}$ $(k \geq 1)$ *is conjugate to* $C^*(\zeta^{k-1/2,+})$ *by* β^L*;*
(5) *A right coideal of type* $A'_{\ell,k}$ $(3 \leq \ell \leq \infty)$ *with* $1 \leq k \leq (\ell+1)/2$ *is conjugate to* $C^*(\zeta^{k-1/2,+}, \zeta^{\ell-k+1/2,-})$ *by* β^L*.*

The spectral patterns of type $A'_{\ell,k}$ are described as follows.

•$A'_{\ell,k}$ $(3 \leq \ell < \infty,\ k \leq \ell/2)$:

$$\bigoplus_{r\in\mathbb{Z}_{\geq 0}} \left(1 + 2\left[\frac{r}{\ell}\right]\right)\pi_r \oplus \bigoplus_{r\in\mathbb{Z}_{\geq 0}} \left(2\left[\frac{r}{\ell}\right] + 1 + \delta_r\right)\pi_{k-1/2+r},$$

•$A'_{\ell,(\ell+1)/2}$ $(3 \leq \ell < \infty$, odd $\ell)$:

$$\bigoplus_{r\in\mathbb{Z}_{\geq 0}} \left(1 + 2\left[\frac{r}{\ell}\right]\right)\pi_r \oplus \bigoplus_{r\in\mathbb{Z}_{\geq 0}} 2\left[\frac{2r+\ell+1}{2\ell}\right]\pi_{r+1/2},$$

where

$$\delta_r = \begin{cases} 0 & \text{if } r - \ell[r/\ell] < \ell - 2k + 1, \\ 1 & \text{if } r - \ell[r/\ell] \geq \ell - 2k + 1. \end{cases}$$

References

[T] R. Tomatsu, Compact quantum ergodic systems, J. Funct. Anal. **254** (2008), no. 1, 1–83.

Department of Mathematics, Hokkaido University, Hokkaido, 060-0810, JAPAN

E-mail address: `tomatsu@math.sci.hokudai.ac.jp`